An Euler-type version of the local Steiner formula for convex bodies

Stefan Glasauer
Fachhochschule Augsburg
Baumgartnerstr. 16, 86161 Augsburg, Germany
stefan.glasauer(ät)fh-augsburg.de

Source: Bulletin of the London Mathematical Society 30 (1998), 618 - 622

1991 Mathematics Subject Classifications: 52A20, 52A38, 52A55

The purpose of this note is to establish a new version of the local Steiner formula and to give an application to convex bodies of constant width. This variant of the Steiner formula generalizes results of Hann [3] and Hug [6], who use much less elementary techniques than the methods of this paper. In fact, Hann [4] asked for a simpler proof of these results ([4], Problem 2 on p. 900). We remark that our formula can be considered as a Euclidean analogue of a spherical result proved in [2], p. 46, and that our method can also be applied in hyperbolic space.

For some remarks on related formulas in certain two-dimensional Minkowski spaces, see Hann [5], p. 363.

For further information about the notions we use below we refer to Schneider's book [9]. Let $\mathcal{K}^n$ be the set of all convex bodies in Euclidean space $\mathbb{R}^n$, i.e. the set of all compact, convex, non-empty subsets of $\mathbb{R}^n$. Let S^{n-1} be the unit sphere. For $K \in \mathcal{K}^n$, let Nor K be the set of all support elements of K, i.e. the pairs $(x, u) \in \mathbb{R}^n \times S^{n-1}$ such that x is a boundary point of K and u is an outer unit normal vector of K at the point x. The support measures (or generalized curvature measures) of K, denoted by $\Theta_0(K, \cdot), \ldots, \Theta_{n-1}(K, \cdot)$, are the unique Borel measures on $\mathbb{R}^n \times S^{n-1}$ that are concentrated on Nor K and satisfy

$$\int\limits_{\mathbb{R}^n \setminus K} f \, d\lambda = \sum_{i=0}^{n-1} \binom{n-1}{i} \int\limits_0^\infty t^{n-i-1} \int\limits_{\mathbb{R}^n \times S^{n-1}} f(x+tu) \, d\Theta_i(K, (x, u)) \, dt \qquad (1)$$

for all integrable functions $f : \mathbb{R}^n \to \mathbb{R}$; here λ denotes the Lebesgue measure on $\mathbb{R}^n$. Equation (1), which is a consequence and a slight generalization of Theorem 4.2.1 in Schneider [9], is called the local Steiner formula. Our main result is the following:

Theorem 1. *Let $K \in \mathcal{K}^n$ be a convex body. Then*

$$\int\limits_{\mathbb{R}^n} f\,d\lambda \;=\; \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\int\limits_0^\infty t^{n-i-1}\int\limits_{\mathbb{R}^n\times S^{n-1}} f(x-tu)\,d\Theta_i(K,(x,u))\,dt \\ + (-1)^n\int\limits_K f\,d\lambda \tag{2}$$

for all integrable functions $f:\mathbb{R}^n\to\mathbb{R}$ for which each term in this sum is finite.

The example of a ball shows that the integrals on the right side of (2) can be infinite for Lebesgue integrable $f \geq 0$, but they are certainly finite if f is bounded and has compact support.

If we let f be the indicator function of K, the above formula reduces to Theorem 1.3 in Hug [6]. For polytopes and smooth bodies, this specialized formula is due to Hann [3].

We remark that (1) and (2) imply that

$$(1-(-1)^n)\int\limits_{\mathbb{R}^n} f\,d\lambda = \sum_{i=0}^{n-1}\binom{n-1}{i}\int\limits_{-\infty}^{\infty} t^{n-i-1}\int\limits_{\mathbb{R}^n\times S^{n-1}} f(x+tu)\,d\Theta_i(K,(x,u))\,dt$$

for all integrable f for which the sum is well-defined.

Let us first state an auxiliary proposition. For a convex polytope P and $i \in \{0,\dots,n\}$, denote by $\mathcal{F}_i(P)$ the set of all i-dimensional faces of P, and by $\mathcal{F}(P)$ the set of all faces of P. For $F \in \mathcal{F}(P)$, let $N(P,F)$ be the normal cone of P at F, i.e. the set of all outer normal vectors of P at the face F (we let $N(P,P) = N(P,\emptyset) := \{0\}$). For subsets $A, B \subset \mathbb{R}^n$, let $A-B$ be the set of all vectors $v-w$ with $v \in A$ and $w \in B$. The indicator function of a set A is written as $\mathbf{1}_A$.

Lemma. *Let P be a convex polytope. Then the map*

$$\sum_{F\in\mathcal{F}(P)\setminus\{\emptyset\}} (-1)^{\dim F}\;\mathbf{1}_{F-N(P,F)}$$

is equal to 1 *almost everywhere on $\mathbb{R}^n$.*

The idea of the proof originates from McMullen [8], and a related result was briefly mentioned in [7], p. 249; see also [2], pp. 42-48.

Proof of the Lemma: Denote the map by α. Define

$$\mathcal{A} := \{F - N(P,F) : F \in \mathcal{F}(P)\setminus\{\emptyset\}\},$$

so every element of $\mathcal{A}$ is an n-dimensional polyhedral set. Let

$$X := \mathbb{R}^n \Big\backslash \bigcup_{C\in\mathcal{A}} \bigcup_{F\in\mathcal{F}_{n-2}(C)} F\,.$$

The complement of X is a set of λ-measure zero. We are going to show $\alpha(x) = 1$ for all $x \in X$.

Fix a vertex v of P and let u be a vector from the interior of $N(P, \{v\})$. There exists a $\lambda > 0$ with the property that $\{v\} - N(P, \{v\})$ is the only set in $\mathcal{A}$ that contains the point $y := v - \lambda u$, since otherwise we get a contradiction to the fact that the interior of $N(P, \{v\})$ is disjoint from all $N(P, F)$, $F \neq \{v\}$. Thus we have $y \in X$ and $\alpha(y) = 1$.

Now let $x \in X$ be arbitrary. Since X is arcwise connected, we can choose a curve $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We shall show that for every facet H of an element of $\mathcal{A}$ and every $s \in [0, 1]$ with $\gamma(s) \in H$, we have

$$\lim_{t\to s} \alpha \circ \gamma(t) = \alpha \circ \gamma(s)\,.$$

From this it will follow that $\alpha \circ \gamma$ is constant and thus $\alpha(x) = 1$, as asserted.

So let $F \in \mathcal{F}(P)\backslash\{\emptyset\}$ and let H be a facet of $F - N(P, F)$. We shall show that there is one and only one $G \in \mathcal{F}(P)\backslash\{\emptyset, F\}$ such that H is a facet of $G - N(P, G)$. The dimensions of F and G differ by 1, and both $F - N(P, F)$ and $G - N(P, G)$ lie on the same side of H. Once all this is shown, it is clear that $\alpha \circ \gamma$ is constant, and the proof is complete.

In order to prove these assertions, let us assume that there is a facet J of $N(P, F)$ with $H = F - J$ (if this is not the case, then there is a facet G of F with $H = G - N(P, F)$, and this case can be treated analogously). There is a $G \in \mathcal{F}(P)$ with $N(P, G) = J$. Note that F is a facet of G. Both $F - N(P, F)$ and $G - N(P, G)$ have H as a facet, and they lie on the same side of H; in fact the outer unit normal vector of G at its facet F (in the affine hull of G) is an outer normal vector of both $F - N(P, F)$ and $G - N(P, G)$ at H. Now let $G' \in \mathcal{F}(P)\backslash\{\emptyset\}$ be such that $G' - N(P, G')$ has H as a facet. Then G' must contain F and must be contained in G because of the dimensions of F and J. Therefore either $G' = F$ or $G' = G$, as required. ■

Proof of Theorem 1: Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue integrable, and let K be a convex polytope. The map

$$\sum_{F\in\mathcal{F}(K)\backslash\{\emptyset\}} (-1)^{\dim F} f\, \mathbf{1}_{F-N(K,F)}$$

coincides with f almost everywhere according to the Lemma. Let $i \in \{0, \ldots, n-1\}$. Using the Fubini theorem, spherical polar coordinates and the explicit representation of the generalized curvature measures in the case of polytopes (see [9], formula

(4.2.2)), we get

$$\sum_{F\in\mathcal{F}_i(K)} \int_{F-N(K,F)} f\,d\lambda = \sum_{F\in\mathcal{F}_i(K)} \int_0^\infty \int_F \int_{N(K,F)\cap S^{n-1}} t^{n-i-1} f(x-tu)\,d\mathcal{H}^{n-i-1}(u)d\mathcal{H}^i(x)dt$$
$$= \binom{n-1}{i} \int_0^\infty \int_{\mathbb{R}^n\times S^{n-1}} t^{n-i-1} f(x-tu)\,d\Theta_i(K,(x,u))\,dt,$$

where $\mathcal{H}^i$ denotes the i-dimensional Hausdorff measure. We deduce

$$\int_{\mathbb{R}^n} f\,d\lambda + \sum_{\substack{i\in\{1,\dots,n-1\}\\ i \text{ odd}}} \binom{n-1}{i} \int_0^\infty \int_{\mathbb{R}^n\times S^{n-1}} t^{n-i-1} f(x-tu)\,d\Theta_i(K,(x,u))\,dt$$
$$= \sum_{\substack{i\in\{0,\dots,n-1\}\\ i \text{ even}}} \binom{n-1}{i} \int_0^\infty \int_{\mathbb{R}^n\times S^{n-1}} t^{n-i-1} f(x-tu)\,d\Theta_i(K,(x,u))\,dt + (-1)^n \int_K f\,d\lambda\,.$$

If f is continuous with compact support, the weak continuity of the generalized curvature measures and the dominated convergence theorem imply that this equation is valid also for arbitrary convex bodies $K \in \mathcal{K}^n$. We can approximate the indicator function of a compact set by a decreasing sequence of continuous functions with compact support. Consequently, this relation holds true for indicator functions of compact sets, therefore also for indicator functions of Borel sets, and hence also for integrable functions. This completes the proof. ■

As a consequence of Theorem 1, we derive a theorem about convex bodies of constant width, which generalizes a well-known result of Blaschke, Santaló, Dinghas, and Debrunner. We use the area measures of Aleksandrov–Fenchel–Jessen, which are defined by means of $S_j(K,\omega) := \Theta_j(K,\mathbb{R}^n\times\omega)$ for Borel subsets ω of S^{n-1}. Since convex bodies of constant width are strictly convex, the measures $S_j(K,\cdot)$ contain the same information as $\Theta_j(K,\cdot)$ for such K.

Theorem 2. *Let $K \in \mathcal{K}^n$ be of constant width b, let ω be a Borel subset of S^{n-1}. Let $j \in \{0,\dots,n-1\}$. Then*

$$S_j(K,-\omega) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} b^{j-i}\, S_i(K,\omega).$$

The case $\omega = S^{n-1}$ gives the classical result stated, e.g., in [1], p. 66, formula (6.6). Although Theorem 2 can also be obtained by methods of Chakerian and Groemer [1], pp. 66-67, or via approximation by smooth bodies (using Theorem 3.3.1 in Schneider

[9]), the present approach seems to be the most direct one. It can also be used in spherical and hyperbolic space, where the other methods cannot be applied in a straightforward way.

Proof: Let $\eta := \mathbb{R}^n \times (-\omega)$, let $\epsilon > 0$ and let f be the indicator function of the local parallel set $\{x + tu \in \mathbb{R}^n : 0 < t \leq \epsilon, (x,u) \in \eta \cap \mathrm{Nor}\, K\}$. Then the Steiner formula (1) tells us that

$$\int_{\mathbb{R}^n} f \, d\lambda = \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j} \epsilon^{n-j} S_j(K, -\omega)\,.$$

An application of Theorem 1 gives

$$\begin{aligned}
\int_{\mathbb{R}^n} f \, d\lambda &= \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \int_b^{b+\epsilon} t^{n-i-1} \int_{\mathbb{R}^n \times S^{n-1}} f(x - tu)\, d\Theta_i(K, (x,u))\, dt \\
&= \frac{1}{n} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \left((b+\epsilon)^{n-i} - b^{n-i}\right) S_i(K, \omega) \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^i \binom{n}{i} \binom{n-i}{n-j} b^{j-i} \epsilon^{n-j}\, S_i(K, \omega) \\
&= \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^i \binom{n}{j} \binom{j}{i} b^{j-i} \epsilon^{n-j}\, S_i(K, \omega).
\end{aligned}$$

A comparison of the coefficients of ϵ^{n-j} shows the result. ∎

Acknowledgements: I wish to thank Professor Paul Goodey for a useful comment and Professor Rolf Schneider for making reference [8] accessible to me.

Literatur

[1] G.D. Chakerian, H. Groemer, 'Convex bodies of constant width', *Convexity and its applications*, editors P.M. Gruber, J.M. Wills (Birkhäuser, Basel 1983), pp. 49 - 96.

[2] S. Glasauer, *Integralgeometrie konvexer Körper im sphärischen Raum*, Thesis, Univ. Freiburg i. Br. 1995, summary: *Diss. Summ. Math.* 1 (1996) 219 - 226.

[3] K. Hann, 'The average number of normals through a point in a convex body and a related Euler-type identity', *Geom. Dedicata* 48 (1993) 27 - 55.

[4] K. Hann, 'What's the bound on the average number of normals?' *Amer. Math. Monthly* 103 (1996) 897 - 900.

[5] K. Hann, 'Normals in a Minkowski plane', *Geom. Dedicata* 64 (1997) 355 - 364.

[6] D. Hug, 'On the mean number of normals through a point in the interior of a convex body', *Geom. Dedicata* 55 (1995) 319 - 340.

[7] P. McMullen, 'Non-linear angle-sum relations for polyhedral cones and polytopes', *Math. Proc. Camb. Phil. Soc.* 78 (1975) 247 - 261.

[8] P. McMullen, 'On the axiomatic characterization of spherical volume', private communication to R. Schneider, 1981.

[9] R. Schneider, *Convex Bodies: the Brunn–Minkowski Theory* (Cambridge University Press, Cambridge 1993).

Fachhochschule Augsburg
Baumgartnerstr. 16
86161 Augsburg
Germany

E-mail: stefan.glasauer@fh-augsburg.de