Paradoxes of sequences of digits.

In the following we consider only real numbers of the interval [0, 1].

A complete infinite binary tree (CIBT, see Fig. 1) contains all binary representations of real numbers in the interval as paths. Every real number is as well represented by an infinite sequence 0.abc... of digits a, b, c, ..., usually written in a horizontal line, as it is represented by a vertical path in the CIBT.

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Level      CIBT

  0          0.
            / \
  1         0   1
           / \   / \
  2       0  1 0 1
  ...     ............         Fig. 1
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The CIBT is uniquely determined by its structure of nodes as well as by its structure of paths. By definition every binary representation of a real number is represented as a path of the CIBT. Therefore all nodes required for that purpose exist in the CIBT too. On the other hand, there cannot be more nodes because an additional node would belong to an additional binary representation. But then the tree without this additional node would not be complete, contrary to the assumption.

[I] Let x represent a digit, 0 or 1, then we can construct

1) a structure S_1 generated by the set of all terminating paths of the form 0.xxx...xxx000...
2) a structure S_2 generated by the set of all paths of the form 0.xxx...xxx(infinite periodical sequence)
3) a structure S_3 generated by the set of all paths of the form 0.xxx...xxx(infinite nonperiodical sequence)

It follows that also each of these structures, S_1 or S_2 or S_3, generates the complete set of nodes and, therefore, each one is the CIBT that contains the complete set of all infinite paths. As, by construction, the structure does not contain any other paths than those used to construct it, there is an unavoidable ambivalence concerning its paths.

[II] Considering only the outmost right hand path 0.111... of the CIBT, we see that this path is generated by the infinite set of terminating paths of the sequence L

0.000...
0.1000...
0.11000...
0.111000...
...

not containing the limit 0.111... of this sequence. On the other hand, it is impossible to distinguish that path from a path that is generated by the sequence L including its limit 0.111... .

[III] Path 0.111... is said to be the infinite union of intersections of terminating paths of L. If so, then there should be at least two nodes of 0.111... recognizable that are in different terminating paths but not in one single terminating path.

For any pair, A and B, of nodes of the path 0.111..., however, it can be shown that there is no pair of paths, p(A) and p(B), such that $A \in p(A) \wedge B \notin p(A) \wedge A \notin p(B) \wedge B \in p(B)$.

There is no single terminating path equal to 0.111... On the other hand no pair (or multitude) of paths can contribute more nodes than one single path can do.

[IV] Every pair of paths that can be distinguished from each other at a level of the CIBT has been separated by a node. We will denote "paths which can be distinguished" by "lines" in order to exclude the argument (not of interest, but frequently stated) that every path consists of uncountably many paths. The fundamental element of the tree structure is shown in Fig. 2. One line goes in, two lines come out.

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 |
 o
/ \    Fig. 2
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This shows that no separation of lines can occur unless a node "o" facilitates it. A node is nothing but a branching-off of lines (i.e., distinct paths). Therefore there are as many nodes as lines (precisely one line more than nodes because the root node has no incoming line). The number of nodes is countable proving the number of lines countable too. Should there be more paths than lines, then the surplus paths could not be distinguished from one another (or they could not contain the root node). Impossibility of distinction in a CIBT would imply impossibility of distinction in a list (as is used, for example, in Cantor's diagonal argument).