Dark numbers

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Abstract The axiom of infinity postulates that every natural number n has a successor Sn. Unfortunately its creators (Dedekind, Peano, Zermelo) forgot to emphasize that every *individually identifiable* number n has an *individually identifiable* successor Sn. They did not bother to state this property explicitly because it was considered a triviality at their times. Only when numerous contradictions showed up in set theory, to the end that even undefinable "real numbers" had to become accepted, this implicit clause sank into oblivion, and nowadays it is impudently claimed that numbers need not be individually identifiable at all. The present paper shows that, in the framework of set theory, already most natural numbers are not *individually identifiable*; they cannot be given a name, cannot be checked as elements of mappings, and therefore they cannot be applied in mathematics.

Introduction

Arithmetic is based on the potentially infinite sequence of definable natural numbers. Also analysis has no use for the actual infinity of the set ℕ. Limits of sequences are never depending on the complete set of terms but only on the formulas defining or constructing them. The only circumstantial evidence of actual infinity arbitrarily adopted in set theory is supplied in geometry. Without actual infinity, for instance of the set of fractions, there are gaps between the definable points of the real line.

On the other hand, an infinite sequence of well-ordered elements of the completed set ℕ is impossible. "Completed" means that there is an end, like ω following upon all natural numbers or 0 following upon the sequence of all unit fractions. A cursor moving from 1 to 0 must pass a last unit fraction, if they all are well-ordered. A last element however would contradict infinity. Therefore the only feasible way to reconcile completeness and infinity is to refrain from well-order. There are mostly natural numbers which cannot be distinguished from each other. They are inaccessible; we briefly call them dark.

It is unfamiliar and hard for mathematicians trained to believe in completed infinity to imagine a potentially infinite set which is finite without having a last fixed element. When the number n belongs to the set, then also n+1, 2n, n^2 and n^{n^n} belong to the set. Of course with n^{n^n} also n^{n^n}+1 and so on belong to the set. This is the potential infinity, accepted by almost all mathematicians before Bolzano and Cantor. The present paper will show a lot of evidence and plenty of proofs for this kind of numbers and discuss the consequences of their existence.

If the actually infinite set ℕ is to be accepted, then it will unavoidably comprise an actually infinite subset Y of undefinable natural numbers, so-called dark natural numbers. Best evidence is this: Every potentially infinite sequence of defined natural numbers can be analyzed at every index. Mappings of $\mathbb{N}$ cannot be analyzed at every index, step by step completely, which means in linear sets: until the end. If *all* natural numbers have been processed individually in linear order, then necessarily a last one has been processed. This holds for every potentially infinite set: It has always a last element, not a fixed one though. In order to avoid this last element in spite of completed infinity, dark numbers are required. A dark number cannot be treated as an individual but only with the whole set *Y*.

Definition: A natural number is "named" or "addressed" or "identified" or "individually defined" or "instantiated" if it can be communicated, necessarily by a finite amount of information, in the sense of Poincaré [1], such that sender and receiver understand the same and *can link it by a finite initial segment to the origin* 0. All other natural numbers are called dark natural numbers [2].

Communication can occur

- by direct description in the unary system like ||||||| or as many beeps, flashes, or raps,
- by a finite initial segment of natural numbers (1, 2, 3, 4, 5, 6, 7) called a FISON,
- as *n*-ary representation, for instance binary 111 or decimal 7,
- by indirect description like "number of colours of the rainbow" or "number of days per week",
- by other words known to sender and receiver like "seven".

Only when a number n is identified we can use it in mathematical discourse and can determine the trichotomy properties of n and of every multiple kn or power n^k or tetration $^k n$ with respect to every identified number k. $\mathbb{N}_{def}$ contains all defined natural numbers as elements – and nothing else. $\mathbb{N}_{def}$ is a potentially infinite set; therefore henceforth it will be called a collection.

Dark numbers in set theory

Ernst Zermelo explained: If among the numbers β there is no largest one, then they have (according to the second generation principle) a "limit" β', which is following next upon all β [3]. This statement is based upon Cantor's principle that "to any finite or infinite set of elements belongs a certain element which is the element *following next* upon all in the succession" [4].

Georg Cantor himself defined "that ω is the *first* whole number following upon all numbers ν, i.e. which has to be called greater than each of the numbers ν" [5], that however the distance "ω - ν is always equal to ω" [6]. "The totality of *all finite cardinal* numbers ν presents the most obvious example of a transfinite set; we call its cardinal number '*Alef-null*', in symbols $\aleph_0$" [7].

If ω is following *next upon* all natural numbers n then nothing exists in between. If on the other hand between every natural number ν and ω there are always ω or $\aleph_0$ natural numbers, then it is obvious that the numbers here denoted by ν must differ significantly from the numbers here denoted by n.

An explanation suggesting itself would be: The actually infinite set $\mathbb{N}$ of all natural numbers n embraces the potentially infinite collection $\mathbb{N}_{def}$ of definable numbers ν. Then however $\aleph_0$ is the cardinal number of all finite cardinal numbers n and not of the definable cardinal numbers ν.

Dark numbers have been introduced also by Sergeyev[1] [8] who calls the largest natural number grossone ① and puts $\mathbb{N}$ = {1, 2, 3, ..., ① - 1, ①}. Of course the grossone and its predecessors ①, ① - 1, ① - 2, ... and also ①/n for every identified natural number n are not identifiable according to the above definition and belong to the set Y.

Dark natural numbers proved by the sequence of FISONs

According to set theory the set $\mathbb{N}$ of all natural numbers is actually infinite. While potential infinity only requires that for every finite initial segment of natural numbers $\{1, 2, 3, ..., n\}$, abbreviated by FISON F_n, there exists a larger one

$$\forall F_n \, \exists F_m\colon |F_n| < |F_m| \wedge F_n \subset F_m$$

actual infinity exchanges quantifiers and states

$$\exists \mathbb{N} \, \forall F_n\colon |F_n| < |\mathbb{N}| \wedge F_n \subset \mathbb{N} \, .$$

Every FISON F_n ends with a natural number n defined by this very segment, because it connects this number to the origin 1. Every finite union of FISONs is a FISON itself again:

$$\{1\} = \{1\}$$
$$\{1\} \cup \{1, 2\} = \{1, 2\}$$
$$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} = \{1, 2, 3\}$$
$$\dots \, .$$

Because of inclusion monotony, this is true – independent of the set of merged segments. And it is independent of the representation, be it as sets of numbers

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, \dots$$

or as (implicitly indexed) strings of symbols

o, oo, ooo, ...

Theorem   More than finitely many FISONs cannot be merged.

Proof: This becomes clear from the pigeonhole principle and the definition "finite initial segment". If each of the first n positive integers has a *unary representation* in form of a string, like ooooo, that is shorter than n then, by the pigeonhole principle, there must be two different positive integers defined by the same unary representation. Clearly this is absurd.

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[1] Sergeyev's theory, based on the inevitable fact that the natural numbers $\mathbb{N}$ count themselves, $|\mathbb{N}|$ = ①, yields the interesting results that the number of integers $\mathbb{Z}$ is $|\mathbb{Z}|$ = 2·① + 1, and the number of fractions $\mathbb{Q}$ is $|\mathbb{Q}|$ = 2·①^2 + 1.

Same holds in case of $\aleph_0$ finite strings. $\aleph_0$ is a fixed quantity such that $\forall n \in \mathbb{N}$: $n < \aleph_0$. If each one of all $\aleph_0$ positive integers has a *unary representation* in form of a string that is shorter than $\aleph_0$ then, by the pigeonhole principle, there must be two different positive integers defined by the same unary representation. Clearly this is absurd too.

Same holds for FISONs. Since the order of numbers in {1, 2, 3, 4, 5} does not matter, it has the same information content as ooooo. ■

$\mathbb{N}$ has more elements than all F_n. It differs from all F_n because it has a greater cardinal number $\aleph_0$. This however is only possible if $\mathbb{N}$ has at least one element that is not in any F_n. But it turns out that no such element can be found. Therefore the actually infinite set $\mathbb{N}$ must embrace elements which have not been identified. We call them dark natural numbers. Almost all elements of $\mathbb{N}$ are undefinable dark numbers. Note that every F_n can be summed, but $\mathbb{N}$ cannot.

Theorem   Almost all natural numbers are dark and therefore are not identifiable.

Proof: Consider the collection $\mathbb{F} = \{F_n \mid n \in \mathbb{N}\}$ of FISONs F_n (which by definition are identified or, if being dark yet, can become identified) and assume that its union is $\mathbb{N}$.

$\mathbb{N}$ does not change when one of the FISONs is omitted:

$$F_1 \cup F_2 \cup F_3 \cup ... \cup F_{n-1} \cup F_{n+1} \cup F_{n+2} \cup F_{n+3} \cup ... = \mathbb{N} .$$

There is no FISON whose omission would change the result of the union. So every FISON can be omitted; the collection of FISONs which can be omitted *separately* without effect is the collection $\mathbb{F}$ of all existing FISONs. Further those FISONs which are predecessors, i.e., subsets, of the omitted F_n can be omitted too with no effect so that we get for all n

$$F_{n+1} \cup F_{n+2} \cup F_{n+3} \cup ... = \mathbb{N} .$$

Since there is no first FISON that cannot be omitted without effect, we get $\bigcup\{\ \} = \{\ \} = \mathbb{N}$. Of course this result is false because $\mathbb{N}$ is not empty. But the usual explanation, that every FISON can be omitted only as long as there remain larger FISONs, does not hold, because according to Cantor's theorem B [9] every set of ordinal numbers has a first element. And the question whether a set is necessary can be decided in every case. Thus the explanation is false, firstly because a FISON without successor does not exist at all and thus cannot be omitted either, and secondly because we can see that as long as F_n is a proper subset of $\mathbb{N}$, it is neither necessary nor sufficient to be relevant in the union producing $\mathbb{N}$. This holds for all FISONs how large they ever may be. Therefore we must accept the implication: *if* $\mathbb{N}$ were the union of all FISONs, *then* $\mathbb{N}$ would be empty:

$$F_1 \cup F_2 \cup F_3 \cup ... = \mathbb{N} \ \Rightarrow\ \{\ \} = \mathbb{N} . \qquad (1)$$

By contraposition, it follows that $\mathbb{N}$ is not the union of only all FISONs, i.e., of only all identifiable natural numbers. What remains? Numbers that cannot be identified: Dark numbers.

Conclusion: If $\mathbb{N}$ is larger than all unions of FISONs, then it must contain something larger than all FISONs, in fact much larger because

$$\forall F_n: |\mathbb{N} \setminus F_n| = \aleph_0 \,. \qquad (2)$$

Therefore almost all elements of $\mathbb{N}$ are undefinable: dark numbers. ■

Every identified natural number is followed by infinitely many natural numbers including potentially infinitely many dark numbers – any difference between two FISONs – which later on may become identified, and an actually infinite set Y, with $|Y| = \aleph_0$, of dark numbers which will remain dark forever and in every system.

The implication (1) only holds if $\mathbb{N}$ is larger than every FISON, because only then every FISON is too small to be relevant for the union. In case of a purely potentially infinite view there is no fixed set $\mathbb{N}$ but the temporary maximum $\mathbb{N}_{def}$ of an ever increasing sequence of FISONs:

$$\mathbb{N}_{def} = F_1 \cup F_2 \cup F_3 \cup \ldots \,. \qquad (3)$$

In this view *not all* FISONs exist simultaneously, there is always a next one and therefore *not all* FISONs can be omitted by omitting those existing in (3).

All individually definable natural numbers fit into one common finite initial segment. $\mathbb{N}_{def}$ is a potentially infinite collection. It does not matter that $n+1$ is not contained in $\{1, 2, 3, ..., n\}$, because it is contained in $\{1, 2, 3, ..., n, n+1\}$. Further n is a variable not a number. But it can be replaced by any definable natural number.

The set of FISONs cannot have cardinality $\aleph_0$ because (1) every FISON has $\aleph_0$ successors, but two consecutive $\aleph_0$-sets are impossible in the natural order of $\mathbb{N}$, (2) the pigeonhole principle (see above) excludes more than a finite number of distinguishable FISONs, (3) their union does not contain any element missing in all FISONs, and (4) $\mathbb{N}$ cannot be exhausted by individually definable numbers but only collectively by sets, like $\mathbb{N} \setminus \mathbb{N} = \emptyset$.

Dark natural numbers proved by the union of FISONs

All infinitely many FISONs $F_n = \{1, 2, 3, ..., n\}$, when subtracted from $\mathbb{N}$, leave almost all natural numbers there according to (2). What happens when the union $\bigcup\{1, 2, 3, ..., n\}$ is subtracted? Often we hear the false argument that the union of all FISONs is larger than all FISONs. But even then we have only two alternatives:

- Either even this union is not sufficient to exhaust $\mathbb{N}$ completely. Then natural numbers remain which cannot be removed individually. They are dark.

- Or no natural numbers remain. Then the infinite difference of natural numbers can only be removed collectively (by the union of FISONs) but not individually (by any FISON). That means, they are dark.

Anyhow dark numbers are indispensible.

There is another argument claiming that all remainders in (2) are different. But in fact all remainders can differ by at most a finite number of elements, because all FISONs and their differences are finite. An infinite number $|Y|$ must be the same in all remainders if they are actually infinite, so-called $\aleph_0$-sets.

Dark natural numbers proved by intersections of endsegments

Definition: $E_n = \{n, n+1, n+2, n+3, ...\}$ is called the endsegment of $n \in \mathbb{N}$. For every identified number n there is a last identified endsegment, $E_n = \{n, n+1, n+2, n+3, ...\}$, containing it, and a first identified endsegment, $E_{n+1} = \{n+1, n+2, n+3, ...\}$, where n has vanished. However the infinite sequence (E_n) of definable endsegments

$$(E_n) = E_1, E_2, E_3, ...$$

and of their finite intersections

$$E_1, E_1 \cap E_2, E_1 \cap E_2 \cap E_3, ...$$

contains only members of cardinality $\aleph_0$. This effect of every identified endsegment is proved by the system

$$\forall n \in \mathbb{N}_{\text{def}}: \bigcap\{E_1, E_2, E_3, ..., E_n\} = E_n \neq \emptyset \quad \text{where} \quad |E_n| = \aleph_0 \,. \tag{4}$$

Every intersection of a finite set of definable endsegments is not empty but infinite. Even the intersection over the potentially infinite collection of identified endsegments

$$\bigcap\{E_n \mid n \in \mathbb{N}_{\text{def}}\} \subseteq \bigcap\{E_n \mid \bigcap\{E_1, E_2, E_3, ..., E_n\} = \aleph_0\} = \aleph_0 \tag{5}$$

or

$$|\bigcap\{\text{E}(n) \mid \text{E}(n) \text{ appears in (4)}\}| = \aleph_0 \tag{5'}$$

is not empty since it contains not more than the E_n of (4), each of which acts like a filter that, *independent of its position and of the presence of other endsegments*, removes, in all environments, up to n-1 natural numbers from an incoming set and lets pass the infinite rest. Its effect at last position in (4) cannot change when appearing at not-last position in (5). A set has no order anyway.

Every intersection of an actually infinite set of endsegments, however, like the complete set, is empty according to set theory

$$\bigcap\{E_n \mid n \in \mathbb{N}\} = \emptyset \tag{6}$$

because every number will be deleted by at least one endsegment.

An actually infinite set has more elements than all finite sets. So it is a legitimate question to ask what endsegments make the set intersected in (6) actually infinite and the result empty? Since all identified endsegments fail, there must be a difference between the collections of identified endsegments in (5) and of all endsegments in (6). Otherwise the contradiction $\emptyset = \aleph_0$ would follow from (5) and (6). However, it is impossible to find an endsegment that is in (6) but not in (5). Only undefinable, i.e., *dark* endsegments, neither subject to universal quantification by their index or first natural number *n* in (4) nor identifiable in (5), can constitute the endsegments necessary for producing the empty intersection in (6). This is the set of dark endsegments.

Counting down to the empty set by intersecting endsegments can only happen one by one in steps of *one natural number per endsegment* because the definition

$$E_n \setminus \{n\} = E_{n+1} \qquad (7)$$

disallows other mechanisms than the loss of one number per term. We find the bijective function

$$subtract(n) = E_{n+1} \ .$$

But no endsegment can be identified that contributes to reduce the intersection from an infinite set of (4) to the empty set of (6) by deleting the infinite set step by step – if (7) holds without an exception, i.e., if mathematics is valid for all steps.

• Remark: In principle every bijection between infinite sets would have to cover undefinable, dark natural numbers or functions of them and requires to identify them, which of course is impossible. In most cases however that is not so obvious.

• Remark: The loss of $\aleph_0$ elements between every definable endsegment and the empty set can occur only by means of as many dark endsegments. Dark numbers solve the puzzle why

$$\aleph_0 = \lim_{n\to\infty} |E_n| \neq |\lim_{n\to\omega} E_n| = 0 \ .$$

$\lim_{n\to\omega} E_n$ is calculated from (6) whereas $\lim_{n\to\infty} |E_n|$ can only be calculated by means of (5).

Dark natural numbers proved by infinitely many infinite endsegments

According to a theorem of ZFC there are infinitely many infinite endsegments. (Note that ZFC knows only *actual* infinity.) In spite of inclusion monotony their intersection is "proved" to be empty. This is mathematically impossible because a sequence of decreasing *infinite* sets cannot be empty (inclusion monotony!). An infinite set is in all predecessors. It is not the same in all terms; the first terms contain somewhat more than the later terms, but this is not a reason to deny the infinite set being in all terms. Like the sand contained in a not yet empty hourglass has been there over all instances from the beginning, the infinite set has been there from the beginning.

But the present argument does not use the intersection. The infinite set of endsegments exhausts all $\aleph_0$ natural numbers as indices. What remains for the contents of the infinite endsegments?

Two consecutive actually infinite sequences in the natural order of $\mathbb{N}$ are impossible. Therefore there are not $\aleph_0$ actually infinite endsegments. There can only be a potentially infinite set of definable endsegments with infinite contents. The actually infinite set of endsegments contains visible infinite endsegments, dark infinite endsegments, and also dark finite endsegments. It decreases according to (7) until the empty set is reached.

Dark natural numbers proved by descending sequences of ordinals

Every sequence of natural numbers ascending from 0 to ω is actually infinite; it has $\aleph_0$ terms. Every sequence of natural numbers descending from ω to 0 is finite. But there is no largest sequence. So the collection of these sequences is potentially infinite. This follows from the axiom of foundation. But above all it is dictated by the practical impossibility to identify a natural number having actually infinitely many predecessors and to choose it as destination for the leap down from ω

1, 2, 3, 4, 5, 6, 7, ...  ω .

If all were identifiable, then each one could be selected without deciding in advance whether we would continue and in which direction.

As already mentioned above Cantor defined "that ω is the *first* whole number following upon all numbers ν" [5], that however the difference "$\omega - \nu$ is always equal to ω" [6].

According to the first statement we find

$$\{0, 1, 2, 3, ..., \omega\} \setminus \mathbb{N} = \{0, \omega\} .$$

Since no cloudy "empty spaces" but only concrete natural numbers have been subtracted from the well-ordered set, and since, after subtraction, ω has the direct predecessor 0, it must have had a direct predecessor before too. The claim that all natural numbers exist and *can* be applied implies that there exists a state where all natural numbers *have* been applied. Otherwise the claim would be void.

According to Cantor's second statement the distance between every identified number *n* (for dark numbers nothing can be said with respect to distances) and ω is $\aleph_0$

$$\forall n \in \mathbb{N}_{def}: \omega - n = \omega > n .$$

This determines an interval of dark ordinals, an actually infinite set

$$Y \subset (\mathbb{N} \setminus \mathbb{N}_{def})$$

of natural numbers larger than every identified *n* but less than ω.

Let L be the greatest natural number instantiated in a system, then numbers like $L + 1$ or $2L$ or L^{L^L} are also instantiated, but with fading reality. Larger numbers are dark but can become instantiated.

Let M be the greatest natural number ever instantiated during the lifetime of the universe, then also numbers like $M + 1$ or $2M$ or M^{M^M} and similar numbers are also instantiated but with fading reality; all greater numbers are dark and remain so.

Dark fractions

If we for a moment assume, counterfactually, that all natural numbers were visible, then they could be housed in the first column of an infinite matrix (see Fig. 1). All positive fractions, then being visible too, could be housed in such a matrix (see Fig. 2).

1, __, __, __, ...	1/1, 1/2, 1/3, 1/4, ...	1/1, __, __, __, ...
2, __, __, __, ...	2/1, 2/2, 2/3, 2/4, ...	1/2, __, __, __, ...
3, __, __, __, ...	3/1, 3/2, 3/3, 3/4, ...	2/1, __, __, __, ...
4, __, __, __, ...	4/1, 4/2, 4/3, 4/4, ...	1/3, __, __, __, ...
5, __, __, __, ...	5/1, 5/2, 5/3, 5/4, ...	2/2, __, __, __, ...
...	...	... .
Fig. 1	Fig. 2	Fig. 3

By Cantor's formula

$$k = (m + n - 1)(m + n - 2)/2 + m \qquad (8)$$

all these fractions can be brought into a sequence

$$1/1, 1/2, 2/1, 1/3, 2/2, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, ... \qquad (9)$$

which also can be housed in the first column of the infinite matrix (see Fig. 3).

But if we try to transform Fig. 2 into Fig. 3 without forgetting the fractions initially residing in the first column (placing them intermediately there where the processed fractions have been taken from), then we will obtain a sequence of matrices. Note that this sequence is as static as (9) and by no means a so-called super task. Fig. 5 shows the first terms of this sequence following Fig. 2. Each matrix is belonging to that fraction of sequence (9) which is for the first time appearing in the first column. This one and the exchanged fraction are printed bold in the following matrices

1/1, **2/1**, 1/3, 1/4, ...	1/1, **3/1**, 1/3, 1/4, ...	1/1, 3/1, **4/1**, 1/4, ...	1/1, 3/1, 4/1, 1/4, ...
1/2, 2/2, 2/3, 2/4, ...	1/2, 2/2, 2/3, 2/4, ...	1/2, 2/2, 2/3, 2/4, ...	1/2, **5/1**, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ...	**2/1**, 3/2, 3/3, 3/4, ...	2/1, 3/2, 3/3, 3/4, ...	2/1, 3/2, 3/3, 3/4, ...
4/1, 4/2, 4/3, 4/4, ...	4/1, 4/2, 4/3, 4/4, ...	**1/3**, 4/2, 4/3, 4/4, ...	1/3, 4/2, 4/3, 4/4, ...
5/1, 5/2, 5/3, 5/4, ...	5/1, 5/2, 5/3, 5/4, ...	5/1, 5/2, 5/3, 5/4, ...	**2/2**, 5/2, 5/3, 5/4, ...
...	...	...	... .

Fig. 5

Although every fraction contained in (9) will be found in the first column and no fraction appearing in (9) can be found sitting permanently in the other columns, never a free place will occur there. The fractions sitting there are dark.

That also most natural numbers are dark aggravates the result. This result however would not have been observed, although it would have been the same, if the fractions were not gathered in the first column but in an additionally made up zeroth column.

Dark real numbers in the Binary Tree

The complete infinite Binary Tree consists of nodes representing bits (binary digits 0 and 1) which are indexed by non-negative integers and connected by edges such that every node has two and only two child nodes. Node number $2n + 1$ is called the left child of node number n, node number $2n + 2$ is called the right child of node number n.

A path p is a subset of nodes having the indices

$$0 \in p$$

and

$$n \in p \;\Rightarrow\; (2n + 1 \in p \;\text{ or }\; 2n + 2 \in p \;\text{ but not both}) .$$

The set $\{a_k \mid k \in \mathbb{N}_0\}$ of nodes a_k is countable as is shown by the indices of the nodes:

```
Level            Bits                     Nodes

  0               0.                        a0.
                /    \                     /    \
  1            0       1                 a1      a2
              / \     / \               /  \    /  \
  2          0   1   0   1            a3   a4 a5   a6
            /\  /\  /\  /\            /\   /\  /\   /\
  3        0 ...                     a7 ...              .
```

Remove all nodes of finite paths from the complete infinite Binary Tree. What remains? If nothing remains, all paths are countable. If more remains (tails of uncountably many infinite paths), these nodes must have dark indexes.

Dark real numbers between sequence and limit

Every strictly monotonic infinite sequence converging to a real limit does not assume this limit before ω. $\aleph_0$ dark terms are following upon every defined term before ω. They are neither suitable for counting nor for proofs of uncountability.

For diagonalization of the folklore Cantor-list

$\boldsymbol{a}_{11}, a_{12}, a_{13}, a_{14}, a_{15}, ...$
$\boldsymbol{a}_{21}, \boldsymbol{a}_{22}, a_{23}, a_{24}, a_{25}, ...$
$\boldsymbol{a}_{31}, a_{32}, \boldsymbol{a}_{33}, a_{34}, a_{35}, ...$
$\boldsymbol{a}_{41}, a_{42}, a_{43}, \boldsymbol{a}_{44}, a_{45}, ...$
$\boldsymbol{a}_{51}, a_{52}, a_{53}, a_{54}, \boldsymbol{a}_{55}, ...$
... .

only the digit sequences (a_{n1}, a_{n2}, a_{n3}, ..., a_{nn}) between the left edge and the diagonal are used. The list

$\boldsymbol{a}_{11}, 0, 0, 0, 0, ...$
$\boldsymbol{a}_{21}, \boldsymbol{a}_{22}, 0, 0, 0, ...$
$\boldsymbol{a}_{31}, a_{32}, \boldsymbol{a}_{33}, 0, 0, ...$
$\boldsymbol{a}_{41}, a_{42}, a_{43}, \boldsymbol{a}_{44}, 0, ...$
$\boldsymbol{a}_{51}, a_{52}, a_{53}, a_{54}, \boldsymbol{a}_{55}, ...$
... .

yields the same diagonal number. Therefore the diagonal number can contain only as many digits, namely less than $\aleph_0$. A digit sequence with $\aleph_0$ digits would require $\aleph_0$ dark digits in addition.

Dark points on the real axis

What hinders the complete covering of the interval (0, 1] by individually definable intervals [1/(n+1), 1/n]? The interval (0, 1] touches zero such that no point fits in between. But no definable point x of (0, 1] touches 0. All definable points satisfy $[x, 1] \subset (0, 1]$.

If every definable point is in some distance from zero, then all definable points are in some distance from zero. The static character of geometry excludes quantifier tricks. But there is no gap, not nothing touching zero. Therefore something else must be between zero and all defined points: a part free of defined points. Moving from 1 to 0 the cursor encounters, before reaching 0, a last shrinking interval. This is made of dark points. It is the advantage of dark points that they collectively undistinguishable can touch zero. For definable points this could not happen.

Dark unit fractions in the interval (0, 1]

The interval (0, 1] is the union of *all* intervals [1/(n+1), 1/n]

$$\bigcup_{n\in\mathbb{N}}\left[\frac{1}{n+1},\frac{1}{n}\right]=(0,1]\ . \qquad (10)$$

The complete covering of the half-open interval (0, 1] by closed intervals [1/(n+1), 1/n] would not be possible, if *every* interval left infinitely many unit fractions uncovered between 0 and its left endpoint. But all *definable* intervals [1/n, 1] leave actually infinitely many unit fractions between zero and their left endpoint uncovered. For every definable natural number k we have

$$\forall k \in \mathbb{N}: \bigcup_{n=1}^{k} [\frac{1}{n+1}, \frac{1}{n}] \neq (0,1] \qquad \text{or} \qquad \bigcup_{n \in \mathbb{N}_{\text{def}}} [\frac{1}{n+1}, \frac{1}{n}] \neq (0,1] \; . \qquad (11)$$

In (10) no point is missing. But every *identified* number n splits the interval in two parts, [1/n, 1] and the remainder (0, 1/n). It is impossible to have less than $\aleph_0$ individually not identified unit fractions in (0, 1/n) since its measure is never zero

$$\forall n \in \mathbb{N}_{\text{def}}: [1/n, 1] \subset ((0, 1/n) \cup [1/n, 1]) = (0, 1] = \bigcup_{n \in \mathbb{N}} [1/n, 1] \; .$$

The interval (0, 1/n) contains $\aleph_0$ unit fraction and remains uncovered. Therefore in (10) there must be more intervals than in (11). This surplus consist of dark intervals.

It is obvious that all those intervals [1/n, 1] which are leaving $\aleph_0$ unit fractions in (0, 1/n) uncovered, cannot cover (0, 1]. Collectively however always infinitely many more unit fractions, namely all, can be covered. [10]

Briefly: When all unit fractions 1/n are removed from the interval (0, 1] then none remains. This proves the existence of all. If only instantiated unit fractions are removed, then $\aleph_0$ remain. $\aleph_0$ unit fractions cannot be addressed and of course cannot be applied in any mapping.

A simple picture: Let a cursor run from 1 to 0. Every passed accessible unit fraction has infinitely many smaller unit fractions as successors. When the cursor passes 0 all unit fractions have been passed. None remains, not even the infinitely many following upon every accessible unit fraction. They are not accessible. They are dark. This prevents that the last unit fraction passed by the cursor can be determined and put in order.

One could claim that the linearity of the problem requires that the cursor never passes two or more unit fractions at one position. Therefore a last one must have been passed. But we don't know about the structure of dark points. We only have to assume that a unit fraction has 1 as its numerator a natural number as its denominator.

Alas if all unit fractions had $\aleph_0$ smaller unit fractions as successors the cursor could never diminish the number of unit fractions between itself and zero to less than $\aleph_0$ and could never reach zero. Even if every unit fraction had only one successor, the physical movement would be hampered by this philosophical assumption. That is impossible.

When the cursor moves from 0 to 1, what is the first unit fraction 1/n encountered? How many smaller unit fractions populate the interval (0, 1/n)? Why were they not met by the curser before? These questions remain in the darkness of dark points and numbers.

Dark states solve Zeno's paradox

Achilles and the tortoise run a race. The tortoise gets a start and the race begins (state 0). When Achilles reaches this point, the tortoise has advanced further already (state 1). When Achilles reaches that point, the tortoise has advanced again (state 2). And so on (state 3, 4, 5, ...). Since Achilles runs much faster than the tortoise, he will overtake (state ω), but only after infinitely many finitely indexed states of the described kind. Their number must be completed. Otherwise Achilles will not overtake. But there must not be a last visible finitely indexed state. (The last 1000 states Achilles remembers have indices much smaller than ω.) This can only be realized by means of dark states.

According to set theory, all states can be put in bijection with all natural numbers. This is impossible as completeness and well-order require a last mark. The three notions "all" and "infinite" and "well-ordered" do not match. This dilemma can only be solved by refraining from well-order of the dark set Y of states.

The basic relation between infinities

Consider a ruler with all unit fractions between 0 and 1 marked. Two such rulers cannot deviate by any mark: *If* $\mathbb{N}$ *is a completed set*, then $|\mathbb{N}|$ is invariable. But only definable elements n can be treated as individuals. All mappings, including Hilbert's hotel, occur in this potential infinity ∞. Collectively always more, namely all the $\aleph_0$ remaining elements can be treated. Therefore, using $\aleph_0$ as the "elastic" rest, we obtain

$$|\mathbb{N}| = \aleph_0 = n + \aleph_0 = \infty + \aleph_0$$

but

$$|\mathbb{N}| \neq |\mathbb{N}| + 1 .$$

Infinite sets can only then be completed, if all elements are existing such that their number is fixed and invariable.

Cantor's "bijections" cover only the potentially infinite initial segments of the concerned sets. That is the natural explanation why all countable sets have the same cardinal number.

Conclusion

We can remove collectively all natural numbers from $\mathbb{N}$ such that none remains; we cannot repeat the same individually since this would include a last one. Individual processing happens in linear order. If all terms of a sequence have been processed then necessarily a last one has been processed. This undermines the definition of countability. We cannot scrutinize every pair of a bijection, but every pair must be *available* for individually scrutinizing it. Individually scrutinizing however is necessarily a linear process that either ends with a last pair (when nothing remains to be scrutinized) or never ends (when always something remains to be scrutinized). But it is impossible that nothing remains and nevertheless no last pair can be scrutinized.

Literature

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