Transfinite Set Theory Contradicted

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Abstract The three basic features of transfinite set theory are shown to be inconsistent. (1) There is no complete infinite set $\mathbb{N}$ with $\aleph_0$ identifiable natural numbers. (2) Assuming that $\mathbb{N}$ exists there is no bijection between countable sets like $\mathbb{N}$ and $\mathbb{Q}$. (3) Even if countability is assumed there are no uncountable sets.

1. There is no actually infinite set $\mathbb{N}$.

The sequence of all Finite Initial Segments Of Natural numbers (FISONs) is represented below

{1}
{1, 2}
{1, 2, 3}
{1, 2, 3, 4}
{1, 2, 3, 4, 5}
... .

The terms of the sequence contain, as elements of FISONs, not more than any finite number of numbers. An infinite union $\mathbb{N}$ of $\aleph_0$ natural numbers is not produced because every FISON and every union of FISONs is finite. The claim that an infinite union does exist and is larger than every FISON can be dismissed by the fact that, according to the pigeonhole principle, there cannot be more FISONs than FISONs have elements. This is best seen in unary representation

o
oo
ooo
oooo
ooooo
... .

Since {1, 2, 3, ..., n} does not depend on the ordering of the elements {1, 2, 3, 4, 5} has same information content as ooooo. There is no exemption from the pigeonhole principle for infinite sets if all elements shall be distinct.

If we assume that an actually infinite set $\mathbb{N}$ exists with, for every $n \in \mathbb{N}$, $|\mathbb{N}| = \aleph_0 > n$, then it is clear that $\forall n \in \mathbb{N}$: $|\mathbb{N} \setminus \{1, 2, 3, ..., n\}| = \aleph_0$ is incompatible with $\mathbb{N} \setminus \{1, 2, 3, ...\} = \emptyset$.

Every FISON $F_n = \{1, 2, 3, ..., n\}$ is the union of its predecessor $F_{n-1} = \{1, 2, 3, ..., n-1\}$ and $\{n\}$. In this way we accumulate ω or $\aleph_0$ finite unions of the form

$$(... ((\{1\} \cup \{2\}) \cup \{3\}) \cup ... \cup \{n\}) = \{1, 2, 3, ..., n\}$$

none of which yields the set $\mathbb{N}$ of all natural numbers although all natural numbers are present as elements in the set of all FISONs created in this way. But if we merge all these FISONs for another time, i.e., all the unsuccessful attempts to establish $\mathbb{N}$ (which is also the set of all last elements of the FISONs), then we get

$$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup ... \cup \{1, 2, 3, ..., n\} \cup ... = \{1, 2, 3, ...\} = \mathbb{N}\ .$$

We execute union number $\omega + 1$ over what already had been merged before (each FISON is in infinitely many unions), and without adding anything further we get a larger set than has been existing before, namely the set of all last elements of the FISONs. The infinite union

$$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup ... = \mathbb{N} \qquad (1)$$

is in the *limit* of the sequence of all finite unions

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, ... \qquad (2)$$

since in both cases all natural numbers are applied. But $\mathbb{N}$ is not contained in (2) or (3):

$$\{1\}, \{1\} \cup \{1, 2\}, \{1\} \cup \{1, 2\} \cup \{1, 2, 3\}, ...\ . \qquad (3)$$

Since from every FISON we know by definition that it is neither sufficient nor necessary to yield the union of FISONs $\mathbb{N}$, we can remove it from the union and find

$$\bigcup\{F_1, F_2, F_3, ...\} = \mathbb{N} \quad \Rightarrow \quad \bigcup\{\ \} = \mathbb{N}\ .$$

This is a highly counterintuitive result[1], at least for those who believe in the union resulting in $\mathbb{N}$. But it becomes easily understandable and clearly obvious without any induction in the form

$$\forall n \in \mathbb{N}: \bigcup\{F_1, F_2, F_3, ..., F_n\} = \mathbb{N} \quad \Rightarrow \quad \{\ \} = \mathbb{N}$$

like

Butter & Bread & Beef = $\mathbb{N}$ $\Rightarrow$ $\{\ \} = \mathbb{N}$.

The assumption of an actually infinite complete set $\mathbb{N}$ implies also complete endsegments.

[1] Usually the handwaving claim is: "But infinitely many FISONs have to remain!"

Definition: $E_n = (n, n+1, n+2, n+3, ...) = \{k \in \mathbb{N} \mid k \geq n\}$ is called the endsegment of $n \in \mathbb{N}$.

Every endsegment has $\aleph_0$ natural numbers as elements. This can be shown by induction: $E_1 = \mathbb{N}$ has cardinality $\aleph_0$. If $|E_n| = \aleph_0$ then $|E_{n+1}| = \aleph_0 - 1 = \aleph_0$. Every endsegment is the finite intersection of itself and of all its predecessors, and every finite intersection has cardinality $\aleph_0$:

$$|E_1| = \aleph_0,\ |E_1 \cap E_2| = \aleph_0,\ |E_1 \cap E_2 \cap E_3| = \aleph_0,\ ... .$$

For every natural number n there is a first endsegment E_{n+1} not containing it. Therefore, according to set theory, the intersection of all endsegments is empty (like the union of all FISONs is $\mathbb{N}$) although there is no empty endsegment.

This result is not acceptable. No natural number can be found[1] that is in any E_n but not in $E_1 = \mathbb{N}$. Inclusion monotony then shows that all natural numbers of E_n are contained in all preceding endsegments, and almost all are contained in all succeeding endsegments too.

<u>Theorem</u> Every endsegment E_n has $\aleph_0$ natural numbers in common with all other endsegments.

Proof: For every endsegment E_m with $m < n$ this follows from the definition. For every endsegment E_{n+k} it follows by commutativity of the intersection: Assume that E_n has less than $\aleph_0$ numbers in common with E_{n+k}, then there is an endsegment, namely E_{n+k} , having less than $\aleph_0$ numbers in common with its predecessor E_n. Contradiction. ■

All endsegments E_n which in finite intersections

$$E_1 \cap E_2 \cap E_3 \cap ... \cap E_n \neq \emptyset$$

provably leave non-empty, even infinite, results should, when joined together with no additional participant, change their behaviour and decrease the intersection below all former benchmarks? This is another version of the clear contradiction that all natural numbers that have infinitely many natural numbers beyond them are all natural numbers that have nothing beyond them.

Remark: $\forall n \in \mathbb{N}: E_n \neq \emptyset$ but $\lim_{n\to\infty} E_n = \emptyset$ would be tantamount to $\omega, \omega, \omega, ... \to 0$. Accepting this kind of limit would invalidate all bijections between infinite sets which are so imperative for set theory.

Remark: We do not use $\lim_{n\to\infty} |E_1 \cap E_2 \cap ... \cap E_n| = |\lim_{n\to\infty} E_1 \cap E_2 \cap ... \cap E_n|$ here but only the simple argument: The infinite intersection of endsegments contains only endsegments.

[1] An empty intersection would require an empty endsegment, which has been excluded, or at least two numbers j and k and two endsegments E_m and E_n such that $j \in E_m \wedge j \notin E_n \wedge k \notin E_m \wedge k \in E_n$, which can be excluded by inclusion monotony, $E_n \supseteq E_{n+1}$, of the sequence (E_n).

2. There is no bijection between $\mathbb{N}$ and $\mathbb{Q}$.

All positive fractions

1/1, 1/2, 1/3, 1/4, ...
2/1, 2/2, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ... (4)
4/1, 4/2, 4/3, 4/4, ...
5/1, 5/2, 5/3, 5/4, ...
...

can be indexed by the Cantor function

$$k = (m + n - 1)(m + n - 2)/2 + m$$

which attaches the index k to the fraction m/n by defining the sequence of fractions

1/1, 1/2, 2/1, 1/3, 2/2, 3/1, 1/4, 2/3, ...

When we assume that all indices 1, 2, 3, ... index the first column then we get Cantor's sequence

1/1, __, __, __, ...
1/2, __, __, __, ...
2/1, __, __, __, ...
1/3, __, __, __, ...
2/2, __, __, __, ...
...

However, when we exchange the fractions in (4) such that Cantor's sequence appears in the first column we will never produce an empty place in the matrix. The first term of the sequence of configuration is (4) since 1/1 stays at its place. Then 1/2 and 2/1 are exchanged, then 2/1 and 3/1, then 1/3 and 4/1, then 2/2 and 5/1, and so on. The second to fifth terms of the sequence of configurations with the exchanged fractions indicated in bold are shown here

1/1, **2/1**, 1/3, 1/4, ...	1/1, **3/1**, 1/3, 1/4, ...	1/1, 3/1, **4/1**, 1/4, ...	1/1, 3/1, 4/1, 1/4, ...
1/2, 2/2, 2/3, 2/4, ...	1/2, 2/2, 2/3, 2/4, ...	1/2, 2/2, 2/3, 2/4, ...	1/2, **5/1**, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ...	**2/1**, 3/2, 3/3, 3/4, ...	2/1, 3/2, 3/3, 3/4, ...	2/1, 3/2, 3/3, 3/4, ...
4/1, 4/2, 4/3, 4/4, ...	4/1, 4/2, 4/3, 4/4, ...	**1/3**, 4/2, 4/3, 4/4, ...	1/3, 4/2, 4/3, 4/4, ...
5/1, 5/2, 5/3, 5/4, ...	5/1, 5/2, 5/3, 5/4, ...	5/1, 5/2, 5/3, 5/4, ...	**2/2**, 5/2, 5/3, 5/4, ...
...	...	...	... .

The crucial point is this: As long as transpositions can be defined, the whole matrix is completely filled. There is not any place without a fraction. Every definable fraction can be found in the first column. Every fraction found in the other columns will not reside there permanently. But never a place will become empty. Of course never a place will be occupied by a "fraction" ∞. All fractions q which can appear at any place of the matrix satisfy $0 < q < \infty$ and have been residing in the matrix from the start.

3. There are no uncountable sets like ℝ or $\mathcal{P}(\mathbb{N})$

The complete infinite Binary Tree consists of nodes representing bits (binary digits 0 and 1) which are indexed by non-negative integers and connected by edges such that every node has two and only two child nodes. Node number $2n + 1$ is called the left child of node number n, node number $2n + 2$ is called the right child of node number n.

The set $\{a_k \mid k \in \mathbb{N}_0\}$ of nodes a_k is countable as shown by the indices of the nodes:

```
Level            Bits                   Nodes

  0               0.                     a_0.
                 /  \                    /  \
  1             0    1                 a_1    a_2
               / \  / \               / \     / \
  2           0  1  0  1            a_3  a_4 a_5  a_6
             /\ /\ /\ /\            /\   /\   /\   /\
  3         0 ...                  a_7 ... .
```

A path p is a subset of nodes with

$$0 \in p$$

and

$$n \in p \;\Rightarrow\; (2n + 1 \in p \;\text{ or }\; 2n + 2 \in p \;\text{ but not both}).$$

The complete infinite Binary Tree contains, as its paths, i.e., as infinite bit strings (a_k) all real numbers between 0.000... and 0.111... .

The Binary Tree containing only all terminating paths is, as far as nodes and edges are concerned, identical with the complete infinite Binary Tree. But how can the paths representing periodic and irrational strings be inserted into the Binary Tree to get the complete infinite Binary Tree? Not at all! 1/3 for instance has no binary representation but is only the *limit* of the sequence of partial sums 0.01, 0.0101, 0.010101, ..., i.e., the series represented by the string 0.010101... related to the path (0., 0, 1, 0, 1, 0, 1, ...). Periodic rational numbers and irrational numbers cannot be represented by strings or paths in the complete infinite Binary Tree.

A countable set can be constructed by the well-known technique, namely using always half of the remaining time for the next step. According to set theory an uncountable set cannot be constructed such that uncountably many elements can be distinguished. So it is possible to construct ℕ and with it all its subsets. But these subsets cannot be distinguished unless it is indicated which elements are to combine. Therefore we find in the Binary Tree with its countably many nodes and uncountably many paths:

- The Binary Tree can be constructed because it consists of countably many nodes and edges.
- The Binary Tree cannot be constructed because it consists of uncountably many *distinct* paths.

The basic structure of the Binary Tree is the branching at a node o

```
     |
     o
    / \
```

where the number 2 of edges leaving a node is equal to the number of 1 incoming edge plus 1, represented by the node: 1 + 1 = 2 .

All paths which can be distinguished on a certain level are distinguished by nodes (or edges). Therefore the number of distinguishable paths grows with the number of nodes. Every node increases the number of distinguishable paths by 1. The number of distinguishable paths is identical to the number of nodes + 1. It could be made equal to it by an additional pre-rootnode o.

The number of all incoming distinguishable paths at some level plus the number of nodes on this level is the number of distinguishable paths leaving this level.

Even "in the infinite", should it exist, a path cannot branch into two paths without a node; the branching creates the node, because a node is defined as a branching point. No increase in distinguishable paths is possible without the same increase in nodes.

Not necessary to mention, at every level the cross-section of the Binary Tree, i.e., the number of nodes at that level, is finite. And, as an upper estimate: even lining up all $\aleph_0$ nodes on a single level would limit the set of paths to $2\aleph_0$.

Finally let us consider a variant of the construction by infinite paths, the game "Conquer the Binary Tree" that only can be lost if set theory is true: You start with one cent. For a cent you can buy an infinite path of your choice in the Binary Tree. For every node covered by this path you will get a cent. For every cent you can buy another path of your choice. For every node covered by this path (and not yet covered by previously chosen paths) you will get a cent. For every cent you can buy another path. And so on. If there are only countably many nodes yielding as many cents but uncountably many paths requiring as many cents, the player will get bankrupt before all paths are conquered. If no player gets bankrupt, the number of paths cannot surpass the number of nodes.

The power set of $\mathbb{N}$, the set $\mathcal{P}(\mathbb{N})$, is a representation of the real numbers of the interval $[0, 1) \subset \mathbb{R}$ in binaries.[1] Therefore it shares its fate.

There is a counter-argument: For every attempted mapping of the subsets of $\mathbb{N}$, the set *M* of those natural numbers which are mapped on sets of natural numbers not containing themselves cannot get indexed itself. If *m* is mapped on *M*, and *m* is a member of *M*, then *m* does not belong to *M*. But when *m* is removed, then *m* must be a member of *M*.

This argument fails because it requires the complete existence of the mapping to define the members of *M*. But complete mappings between infinite sets cannot exist.

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[1] For instance { } represents 0.0..., {1} represents {0.1000...}, {1, 3, 5, ...} represents 0.101010... = 2/3.