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INFLUENCE OF VOLUME AND STRESS DISTRIBUTION ON THE SHEAR STRENGTH AND TENSILE STRENGTH PERPENDICULAR TO GRAIN

by

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1. ABSTRACT

In 1939 WEIBULL [1] developed a theory, which allows to estimate the influence of the size of the stressed volume and the stress-distribution over this volume on the strength of homogeneous, isotropic materials with brittle fracture behaviour. Although wood as material is neither homogeneous nor isotropic, it has been shown, that the application of Weibull's theory is possible. Weibull's theory has even entered several design codes (CIB-structural timber design code, Euro-Code 5, Canadian code ...). It is used especially in the case of shear and tension perpendicular to the grain, not least because of the existing brittle fracture behaviour.

In this paper, the application of Weibull's theory in the case of shear and tension perpendicular to the grain is discussed.

2. GENERAL

To estimate the influence of the stress-distribution and the size of the stressed volume on the strength, the two-parameter Weibull-distribution is used. The cumulative frequency is defined by:

$$S = 1 - \exp\left[-\int_V \left(\frac{\sigma}{\sigma'}\right)^k dV\right] \qquad (1)$$

where σ' and k are the parameters of the Weibull-distribution, and $\sigma = \sigma(x,y,z)$ is the stress distribution over the volume V.

The (constant) width W of a beam with rectangular cross section is assumed not to influence its strength, so that the integral in eq (1) may be written [2]

$$\int_V \left(\frac{\sigma}{\sigma'}\right)^k dV = W \cdot \int_{x=o}^{L} \int_{y=o}^{D(x)} \left(\frac{\sigma(x,y)}{\sigma'}\right)^k dy\ dx \qquad (2)$$

For a beam with constant depth (D(x) = D) eq. (2) may be written:

$$\int_V \left(\frac{\sigma}{\sigma'}\right)^k dV = W \cdot \left(\frac{\max\ \sigma}{\sigma'}\right)^k \cdot L \cdot \underbrace{\int_{\varepsilon=o}^{1} f^k(\varepsilon)\ d\varepsilon}_{\lambda_L^k} \cdot D \cdot \underbrace{\int_{\xi=o}^{1} f^k(\xi)\ d\xi}_{\lambda_D^k} \qquad (3)$$

or

$$\int_V \left(\frac{\sigma}{\sigma'}\right)^k dV = V \cdot \left(\left(\frac{\max\ \sigma}{\sigma'}\right) \cdot \lambda_L \cdot \lambda_D\right)^k \qquad (4)$$

where V = stressed volume

max σ = maximum stress occuring over the volume V

$f(\varepsilon)$, $f(\xi)$ = dimensionless stress-distribution over the length and depth resp. related to max σ

λ_L, λ_D = "fullness-parameters" to describe the fullness of the stress-distribution.

λ_L and λ_D are dependant on the stress-distribution and the exponent k of the 2-parameter Weibull-distribution. A value of λ near 1 stands for a nearly constant stress distribution.

In fig.1 equations for the determination of the fullness-parameters λ are given as a function of the exponent k (see also [3]). The equation for the parabolic stress distribution (⑥) is only an approximation, because the integral can not be solved directly with a quadratic stress distribution. The exponent k only depends on the variation of the distribution and may approximately be determined by the following equation [2]:

$$k \cong \frac{1,15}{v} \qquad (5)$$

where v is the coefficient of variation.
In the case of shear and tension perpendicular to the grain, a value of $k \sim 5$ may be assumed, corresponding to a coefficient of variation of $v \sim 0,23$. For $k = 5$ the values of the fullness-parameters λ are shown in fig. 2 and fig. 3 for any parabolic stress distribution. Fig. 2 is used when the maximum stress occurs at an edge, whereas fig. 3 is used, when the maximum stress occurs along the span.

If the fullness-parameter λ_L can not be determined directly by fig. 1-3, due to the kind of loading or support, it is possible to divide the given stress distribution in several fields (with the length l_i), for which the fullness-parameters λ_i can be determined according to fig. 1-3. The fullness-parameter λ_L of the beam may then be determined by:

$$\boxed{\lambda_L^5 = \Sigma \frac{l_i}{L} \cdot \left(\frac{\max \sigma_i}{\max \sigma} \cdot \lambda_{L,i}\right)^5} \qquad (6)$$

where

l_i = length of the i-th field

$\max \sigma_i$ = maximum stress occuring along l_i

L = length of the beam

$\max \sigma$ = maximum stress occuring along L

$\lambda_{L,i}$ = fullness-parameter of the i-th field

Eq. (8) may therefore be written as

$$\boxed{\frac{\max\tau_2}{\max\tau_1} = \frac{k_{L/D,1}}{k_{L/D,2}} \cdot \frac{\lambda_{L,1}}{\lambda_{L,2}} \cdot \frac{\lambda_{D,1}}{\lambda_{D,2}} \cdot \left(\frac{V_1}{V_2}\right)^{0,2}} \tag{10}$$

This relationship is valid for beams with rectangular cross section and constant depth.

3.2 Shear stress in tapered beams

In the case of a tapered beam, the depth D is not a constant (cf.eq.(2) and eq.(3)).

With the substitution $\xi = y/D(x)$ eq. (2) may be written

$$\int\left(\frac{\tau}{\tau'}\right)^k dV = W\cdot\left(\frac{\max\tau}{\tau'}\right)^k \cdot L \cdot \underbrace{\int_{\varepsilon=0}^{1} f^k(\varepsilon)\cdot h(\varepsilon)d\varepsilon}_{\lambda^k_{L,tap}} \cdot D_{\max\tau} \cdot \underbrace{\int_{\xi=0}^{1} f^k(\xi)\, d\xi}_{\lambda^k_D} \tag{11}$$

or

$$\int\left(\frac{\tau}{\tau'}\right)^k dV = W\cdot L\cdot D_{\max\tau} \cdot \left(\frac{\max\tau}{\tau'} \cdot \lambda_{L,tap}\cdot\lambda_D\right)^k \tag{12}$$

where

$D_{\max\tau}$ = depth of the beam where $\max\tau$ occurs

$\lambda_{L,tap}$ = fullness-parameter of the stress distribution over the length L, taking into account the variable depth

$h(\varepsilon)$ = dimensionless function of the depth related to $D_{\max\tau}$

In the case of a uniformly distributed load i.e. a linear distribution of the shearforce, the values of $\lambda_{L,tap}$ are given in fig. 6, assuming a value of k = 5 for the exponent of the Weibull-distribution.

According to eq.(6) of section 2, the fullness-parameter $\lambda_{L,tap}$ may be calculated knowing the $\lambda_{L,tap,i}$-values of each field of the beam (with the length l_i):

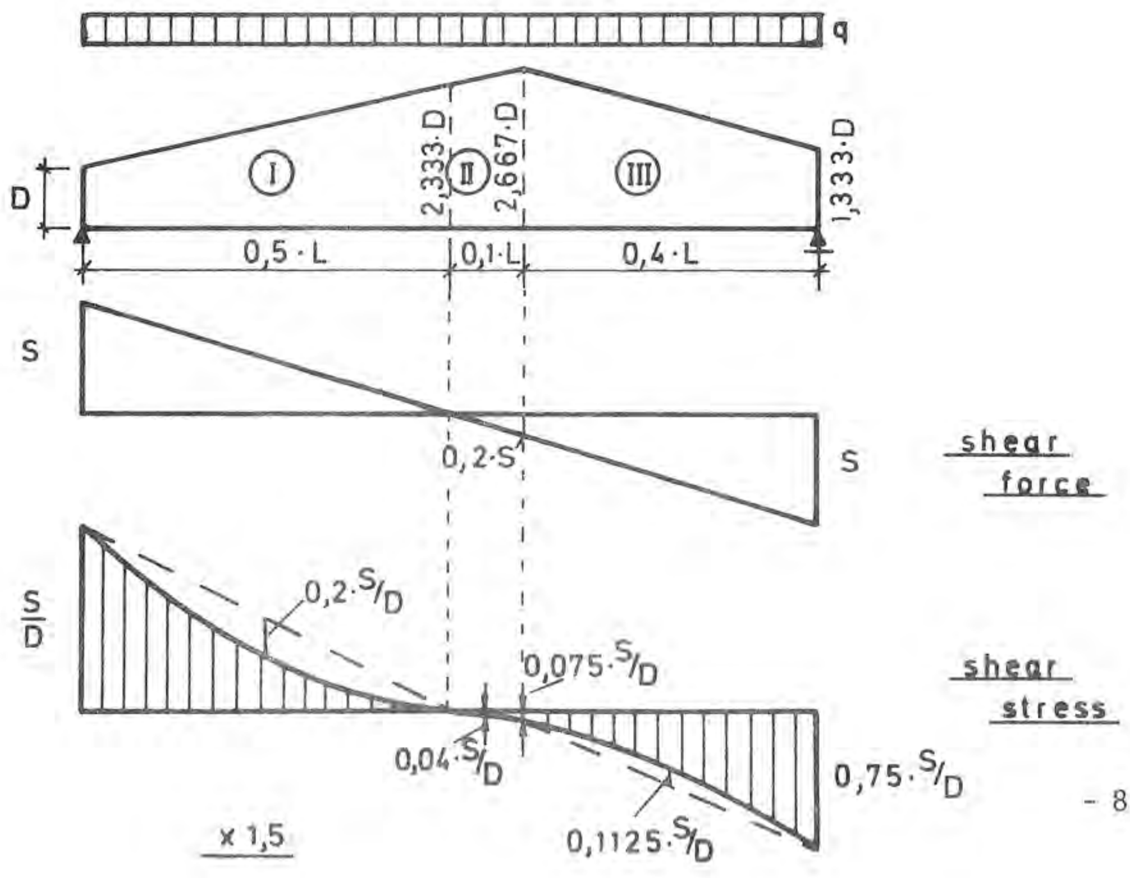
$$\lambda^5_{L,tap} = \Sigma \frac{l_i}{L} \cdot \frac{D_{max\tau_i}}{D_{max\tau}} \cdot \left(\frac{max\tau_i}{max\tau} \cdot \lambda_{L,tap,i}\right)^5 \tag{13}$$

where

$D_{max\tau_i}$ = depth of the beam in the i-th field, where the maximum shear stress max τ_i occurs

$D_{max\tau}$ = depth of the beam, where the maximum shear stress max τ occurs

Example



field	S_1/S_2 ($S_1 < S_2$)	D_1/D_2	$\lambda_{L,tap,i}$ see fig. 6
(I)	0	2,33	0,625
(II)	0	0,875	0,710
(III)	0,2	2,0	0,658

with eq. (13):

$$\lambda^5_{L,tap} = 0,5 \cdot \frac{1}{1} \cdot (\frac{1}{1} \cdot 0,625)^5 + 0,1 \cdot \frac{2,667}{1} (\frac{0,075}{1} \cdot 0,710)^5$$

$$+ 0,4 \cdot \frac{1,333}{1} (\frac{0,75}{1} \cdot 0,658)^5$$

$$= 0,0633$$

$$\lambda_{L,tap} = 0,576 \quad \text{and } V = W \cdot D \cdot L$$

An approximation of this value for $\lambda_{L,tap}$ may be obtained with eq. (6): this calculation is based on the stress-distribution of the tapered beam, where the variable depth is considered in contrast with the distribution of the shear force. This approximation is determined with the help of fig. 2 and eq. (6):

field	τ_1/τ_2	τ_0/τ_2	$\lambda_{L,i}$
(I)	0	- 0,2	0,63
(II)	0	0,033	0,71
(III)	0,1	- 0,15	0,66

$$\lambda_L^{\,5} = 0{,}5\cdot(\tfrac{1}{1}\cdot 0{,}63)^5 + 0{,}1(\tfrac{0{,}075}{1}\cdot 0{,}71)^5 + 0{,}4\cdot(\tfrac{0{,}75}{1}\cdot 0{,}66)^5$$

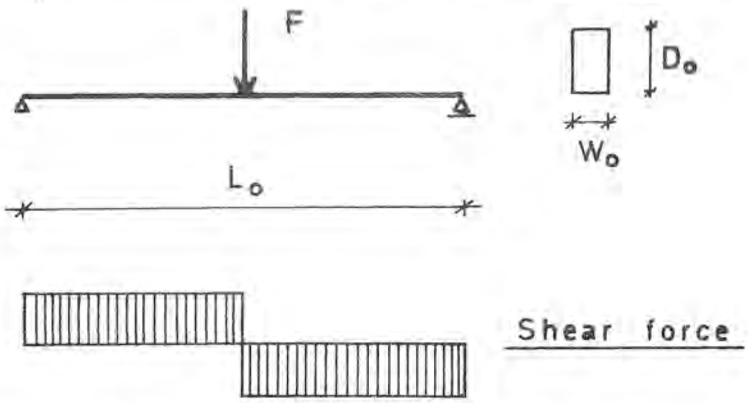
$$= 0{,}0615$$

$$\lambda_L = 0{,}573 \cong 0{,}576 = \lambda_{L,tap}$$

Therefore the fullness-parameters for tapered beams may be calculated in good approximation according to section 2 and section 3.1 resp., based on the distribution of the shear stress, whereas in the case of beams with rectangular cross section and constant depth the fullness-parameters may be calculated on the basis of the distribution of the shear force.

3.3 Design

If the characteristic shear strength $\tau_{0,k}$ is determined using the following test specimen



we have: $\lambda_{L_o} = 1{,}0$ (constant stress distribution)

$\lambda_{D_o} = 0{,}82$ (parabolic stress distribution)

k_{L_o/D_o} according to eq. (9)

and $V_o = W_o\cdot D_o\cdot L_o$.

Assuming a parabolic stress distribution over the depth for all beams, the characteristic shear strength of any beam may be determined by:

$$\tau_k = \frac{1}{\lambda_L} \cdot \frac{k_{L_o/D_o}}{k_{L/D}} \cdot \left(\frac{V_o}{V}\right)^{0,2} \cdot \tau_{o,k} \qquad (14)$$

where λ_L = fullness-parameter according to section 2 (fig. 1-3)

$k_{L/D}$ = determined according to section 3.1 (eq. 9)

$V = W \cdot L \cdot D_{max\tau}$

W = Width of the beam

L = length of the beam

$D_{max\tau}$ = depth of the beam where $max\tau$ occurs

4. Tension perpendicular to the grain

4.1 Comparison of the theory with test results [6]

Kolb/Frech [6] tested three types of beams:
curved beams (type I), cambered beams (type II) and tapered cambered beams (type III). The beam configurations, test set-up and test results are given in fig. 7.

The maximum tensile stress perpendicular to the grain is calculated according to the following equations [7]:

$$max\ \sigma_{\perp} = \kappa \cdot \frac{6\ M}{W \cdot D_{ap}^2} \qquad (15)$$

$$\kappa = A + B \cdot \left(\frac{D_{ap}}{R_{ap}}\right) + C \cdot \left(\frac{D_{ap}}{R_{ap}}\right)^2 \qquad (16)$$

with

$$A = 0{,}2 \cdot \tan\gamma$$

$$B = 0{,}25 - 1{,}5\cdot\tan\gamma + 2{,}6\cdot\tan^2\gamma$$

$$C = 2{,}1\cdot\tan\gamma - 4\cdot\tan^2\gamma$$

γ = slope of the beam (upper border) at the apex (for type I : $\gamma = 0$)

D_{ap} = depth of the beam at the apex

$$R_{ap} = R_1 + D_{ap}/2 = 6{,}0\ \text{m} + \frac{D_{ap}}{2}$$

The tensile strength perpendicular to the grain is shown for all beams in fig. 8. This fig. clearly shows the decrease of the tensile strength perpendicular to the grain within each type with increasing slope γ and the different bearing capacity between the three types of beams: the tensile strength perpendicular to the grain of type I is on average 0,75 times the strength of type II and 0,625 times the strength of type III. It shall now be shown how far these different bearing capacities can be explained by Weibull's theory.

The distribution of the tensile stress perpendicular to the grain was investigated with the method of finite elements. The following beams were considered:

beam I.3, II.3, III.3 } to investigate the differences between the beam types

III.3, III.2, III.1 } to investigate the differences within one beam type

In table 1 the κ-values of eq. (16) are compared with the κ-values, determined by the method of finite elements.

Table 1:

Beam	D_{ap} mm	R_{ap} mm	κ eq.(16)	κ FE
I.3	1000	6500	0,0385	0,0394
II.3	1450	6725	0,0942	0,0934
III.3	1450	6725	0,0942	0,0935
III.2	1250	6625	0,0700	0,0705
III.1	1110	6555	0,0535	0,0534

This comparison shows a good agreement (also for the tapered cambered beams, if the slope of the upper border is used in eq. (16)). The fullness parameters were determined by

$$\lambda_L \cdot \lambda_D = \left[\frac{\Sigma\left(\frac{\sigma_i \perp}{\max \sigma\perp}\right)^5 \cdot V_i}{V_{tot}}\right]^{0,2} \qquad (17)$$

where

V_i = volume of a finite element with tensile stress perp. to grain

$\sigma_i \perp$ = tensile stress perp. to grain of a finite element with volume V_i

V_{tot} = ΣV_i = stressed volume

max $\sigma\perp$ = maximum tensile stress perp. to grain in the volume V_{tot}

- 13 -

These values are given in table 2.

Table 2:

Beam	V_{tot} m^3	$\Sigma(\frac{\sigma_i \perp}{max\sigma\perp})^5 \cdot V_i$ m^3	$\lambda_L \cdot \lambda_D$
I.3	0,922	0,05547	0,570
II.3	0,835	0,05715	0,585
III.3	0,420	0,05880	0,675
III.2	0,310	0,04800	0,689
III.1	0,246	0,03320	0,670

In design calculations the volume V_c of the curved part of the beam(between the points of tangency)is used. This volume corresponds to the shaded area in fig. 7.

According to eq. (7) and eq. (8) resp., the fullness-parameters $(\lambda_L \cdot \lambda_D)_c$ corresponding to the volume V_c are calculated by

$$(\lambda_L \cdot \lambda_D)_c = \lambda_L \cdot \lambda_D \cdot \left(\frac{V_{tot}}{V_c}\right)^{0,2} \qquad (18)$$

and given in table 3.

Table 3:

Beam	V_c m^3	$(\lambda_L \cdot \lambda_D)_c$
I.3	0,545	0,633
II.3	0,667	0,612
III.3	0,446	0,667
III.2	0,287	0,700
III.1	0,147	0,743

The higher fullness-parameters of beam-type III may be explained by the fact, that the whole volume V_c of type III is located between the loading points (constant bending moment) whereas a part of the volumes V_c of type I and II is located outside the loading points (i.e. the stress in this part is lower). Another factor that might explain the higher values of $(\lambda_L \cdot \lambda_D)_c$ for type III is the influence of the loading points: in case of beam type I and II a greater part of the volume V_c is strained by a compressive stress perpendicular to the grain.

For the investigation of the bearing capacity within beam type III, the expected $\frac{\max\sigma_i \perp}{\max\sigma_j \perp}$ -values were calculated according to eq.(8) and compared with the test-values (see table 4).

Table 4:

	$\frac{\sigma_i \perp}{\sigma_j \perp}$					
j \ i	III.1		III.2		III.3	
	eq(8)	test	eq(8)	test	eq(8)	test
III.1	1,0	1,0	0,928	0,914	0,892	0,864
III.2	1,077	1,094	1,0	1,0	0,961	0,945
III.3	1,121	1,158	1,041	1,058	1,0	1,0

This comparison shows a good agreement between the theoretical values according to Weibull's theory and the test values.

The decrease of the bearing capacity with increasing slope γ can thus be explained and numerically evaluated with the help of Weibull's theory.

The expected ratios of the tensile strength perpendicular to the grain for the different beam types are given in table 5 and are again compared with the test results.

Table 5:

	$\frac{\sigma_i\perp}{\sigma_j\perp}$					
j \ i	I.3		II.3		III.3	
	eq.(8)	test	eq.(8)	test	eq.(8)	test
I.3	1,0	1,0	0,994	1,471	0,988	1,770
II.3	1,006	0,680	1,0	1,0	0,994	1,204
III.3	1,012	0,565	1,006	0,831	1,0	1,0

According to Weibull's theory (eq.(8)) all types would have the same tensile strength perp. to the grain: the value $(\lambda_L \cdot \lambda_D)_c \cdot (V)^{0,2}$ (cf. table 3) is nearly a constant for all beam types. The oppositely oriented influences of a greater volume and a lower fullness-parameter counteract, so that the expected strength is the same for all three beam types.

The tests however showed a clear tendency, that the tensile strength perp. to the grain of the cambered beams is higher than the strength of the curved beams with constant depth, and that the strength of the tapered cambered beams is even higher than the strength of the other two beam types.

As too little is known about the tests described in [6], no explanation could be found concerning the contradiction between the theoretical and the test-values.

Therefore further (theoretical and experimental) investigation in this field is required.

4.2 Design

The characteristic tensile strength perpendicular to the grain $\sigma_0\perp$ is determined by a pure tension test (i.e. $\lambda_L \cdot \lambda_D = 1,0$) with a test specimen of volume V_0.

The characteristic strength of any beam can be calculated according to eq. (8):

$$\sigma_k \perp = \frac{1}{\lambda_L \cdot \lambda_D} \cdot \left(\frac{V_o}{V}\right)^{0,2} \cdot \sigma_{o,k} \perp \qquad (19)$$

According to the draft of Eurocode V (October 1985), the characteristic tensile strength perp. to grain of the strength class C3 (determined with a test specimen of volume $V_o = 0{,}02\ m^3$) is:

$$\sigma_k \perp = 0{,}4 \quad N/mm^2$$

In the case of beam I.3 we could expect (according to eq. 19 and table 3) a characteristic strength:

$$\sigma_k \perp = \frac{1}{0{,}633} \cdot \left(\frac{0{,}02}{0{,}545}\right)^{0,2} \cdot 0{,}4 \approx 0{,}33\ N/mm^2$$

Assuming a coefficient of variation of about 25%, the mean strength may be calculated approximately to

$$\sigma \perp \cong 0{,}33 \cdot \frac{1}{1-1{,}645 \cdot 0{,}25} \cong 0{,}56\ N/mm^2$$

(Gauss-distribution)

This value is approximately reached by beam I.3 (mean strength ~ 0,65 N/mm²).

With the assumption, that the decrease of strength with increasing slope γ within one beam type can be explained by Weibull's theory (see section 4.1), eq. (19) may be used for beam type I (curved beam with constant depth).

Theoretically, eq. (19) is also valid for beam type II and III, but the higher bearing capacities of these beams (c.f. fig 8) might be taken into account in the following way:

$$\sigma_k \perp = 1{,}3 \cdot \frac{1}{\lambda_L \cdot \lambda_D} \cdot \left(\frac{V_o}{V}\right)^{0,2} \cdot \sigma_{o,k} \perp \qquad (20)$$

for beam type II (cambered beams)

and

$$\sigma_{k\perp} = 1,6 \cdot \frac{1}{\lambda_L \cdot \lambda_D} \cdot \left(\frac{V_o}{V}\right)^{0,2} \cdot \sigma_{o,k\perp} \qquad (21)$$

for beam type III (tapered cambered beams).

5. SUMMARY AND CONCLUSIONS

Weibull's theory of brittle fracture is used to describe the influence of the stress distribution and the size of the stressed volume on the strength of a beam.

The determination of the so called fullness-parameters λ (which stand for the fullness of the stress-distribution) is shown. Also a mathematical relationship between the expected ratio of the strength of two beams and their fullness-parameters and their stressed volume has been deducted.

The application of this theory and a possible design method has been shown in the case of shear stress and tensile stress perpendicular to the grain.

Because of the differences between the theoretical and the experimental results in the case of tensile stress perpendicular to the grain, further investigations are required. The application of a modified weakest link theory (with weighted influences of the beam-length and depth) as well as the further dependency of the strength on the wood-properties (density, growth rings, knots ...) will probably be investigated in a proposed research program in Karlsruhe.

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[7] Blumer, H. 1972/79: Spannungsberechnungen an anisotropen Kreisbogenscheiben und Satteltträgern konstanter Dicke. Veröffentlichung des Lehrstuhls für Ingenieurholzbau und Baukonstruktionen, Universität Karlsruhe, Germany.

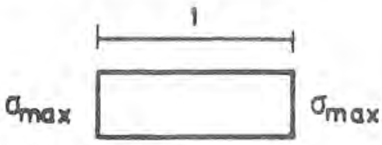
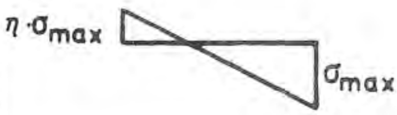
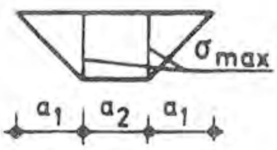
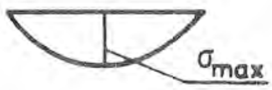
	stress-distribution	$\lambda = \left[\int_{\varepsilon=0}^{1} f^{k}(\varepsilon)\, d\varepsilon \right]^{1/k}$
①		1,0
②		$\left(\frac{1}{k+1} \right)^{1/k}$
③		$\left(\frac{1}{k+1} \cdot \frac{1-\eta^{k+1}}{1-\eta} \right)^{1/k}$
④		$\left(\frac{1}{k+1} \cdot \frac{1+\lvert\eta\rvert^{k+1}}{1+\lvert\eta\rvert} \right)^{1/k}$
⑤		$\left[\frac{1}{k+1} (1 + a_2 \cdot k) \right]^{1/k}$
⑥		$\left[\frac{1}{k+1} (1 + 0{,}345\, k - 0{,}027 \cdot k^2 + 0{,}0013 \cdot k^3) \right]^{1/k}$

fig.1: fullness-parameter λ as a function of the exponent k of the two-parameter Weibull-distribution

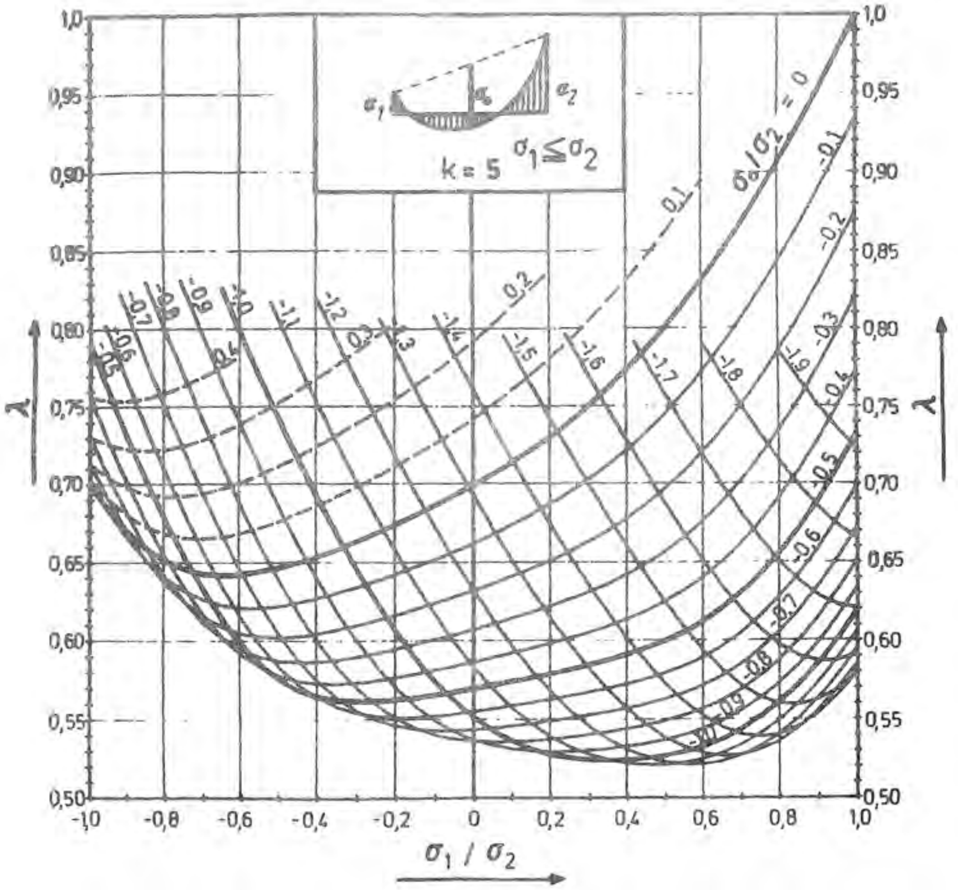


fig.2: fullness-parameter λ for a parabolic stress distribution, with the maximum stress occuring at the edge

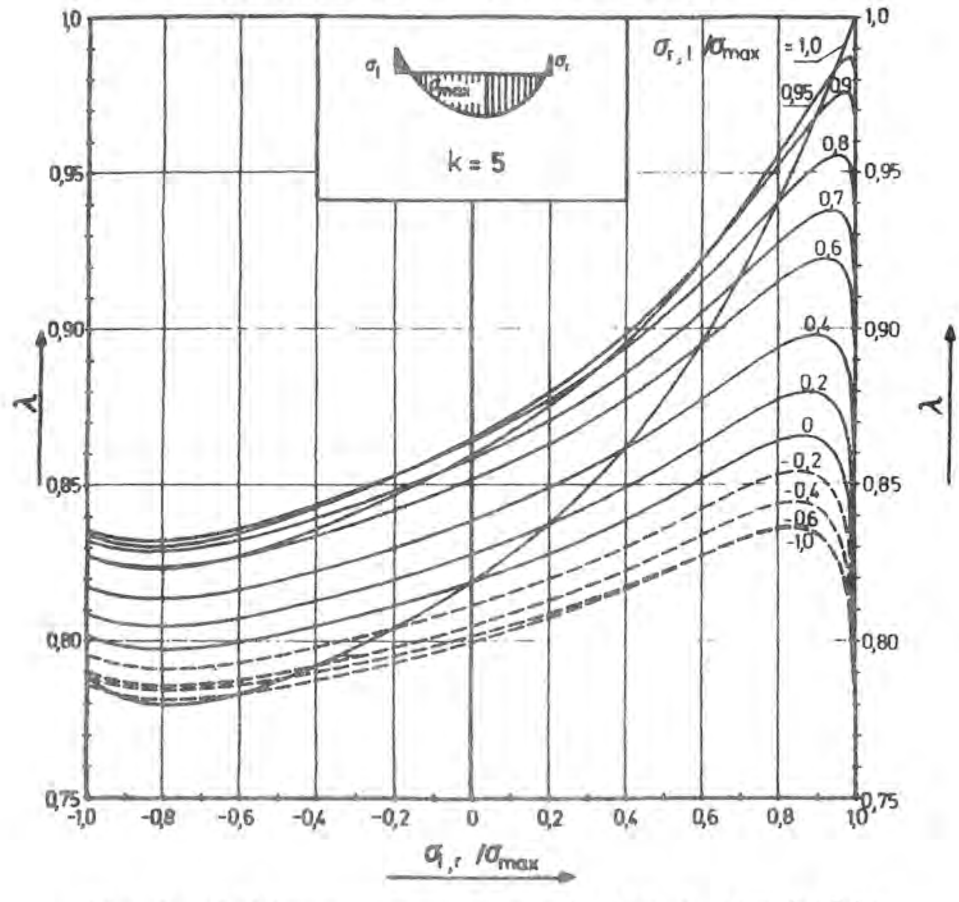


fig.3: fullness-parameter λ for a parabolic stess distribution, with the maximum stress occuring along the span

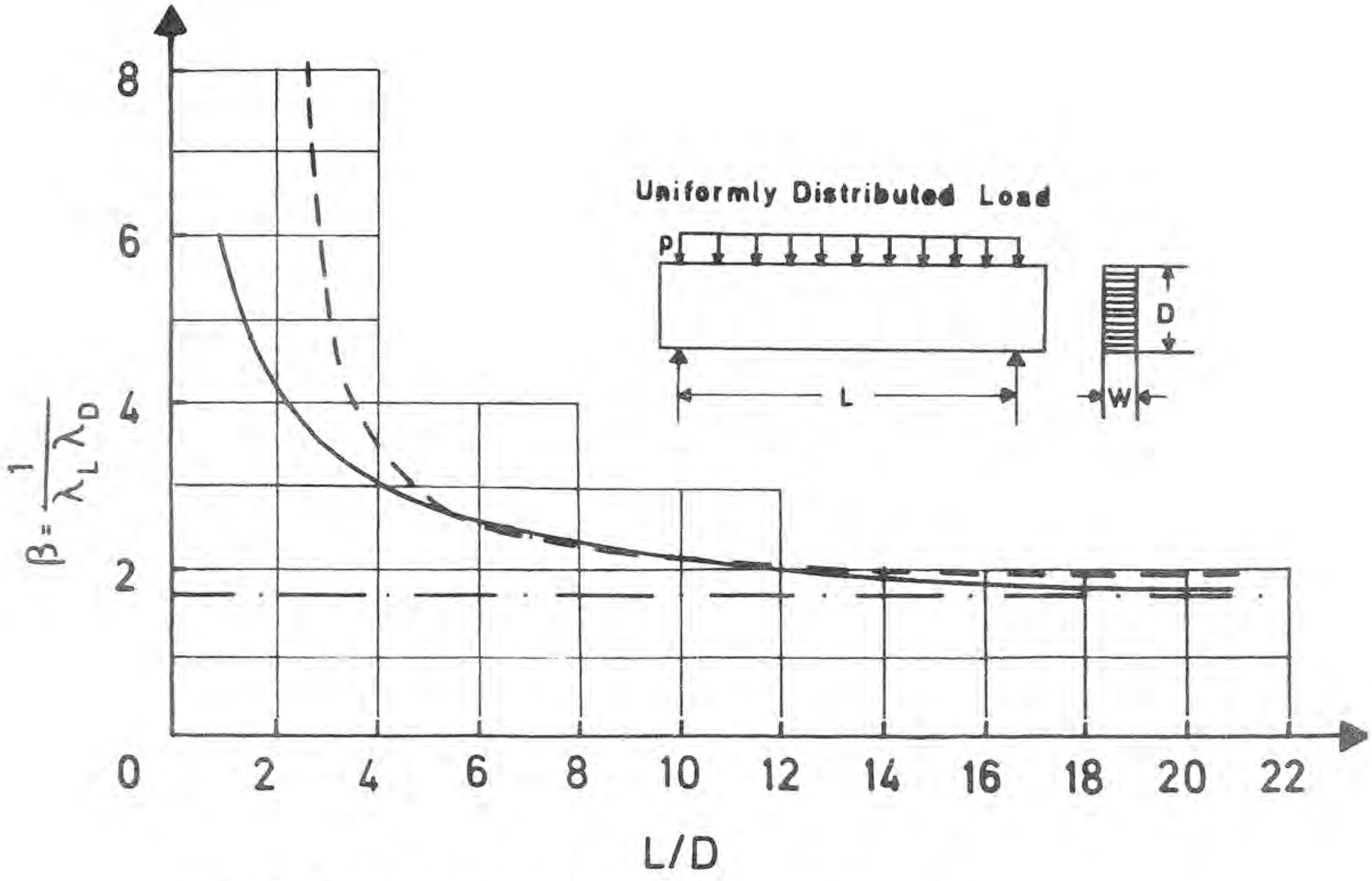


fig.4: β-values for uniformly distributed load

——— according to Foschi/Barrett [4]

—·— according to section 2 ($\frac{1}{\lambda_L \cdot \lambda_D} = \frac{1}{\lambda_{\Delta} \cdot \lambda_D} = \frac{1}{0,70 \cdot 0,82} = 1,742$)

----- $1,742 \cdot \frac{L/D}{L/D-2}$

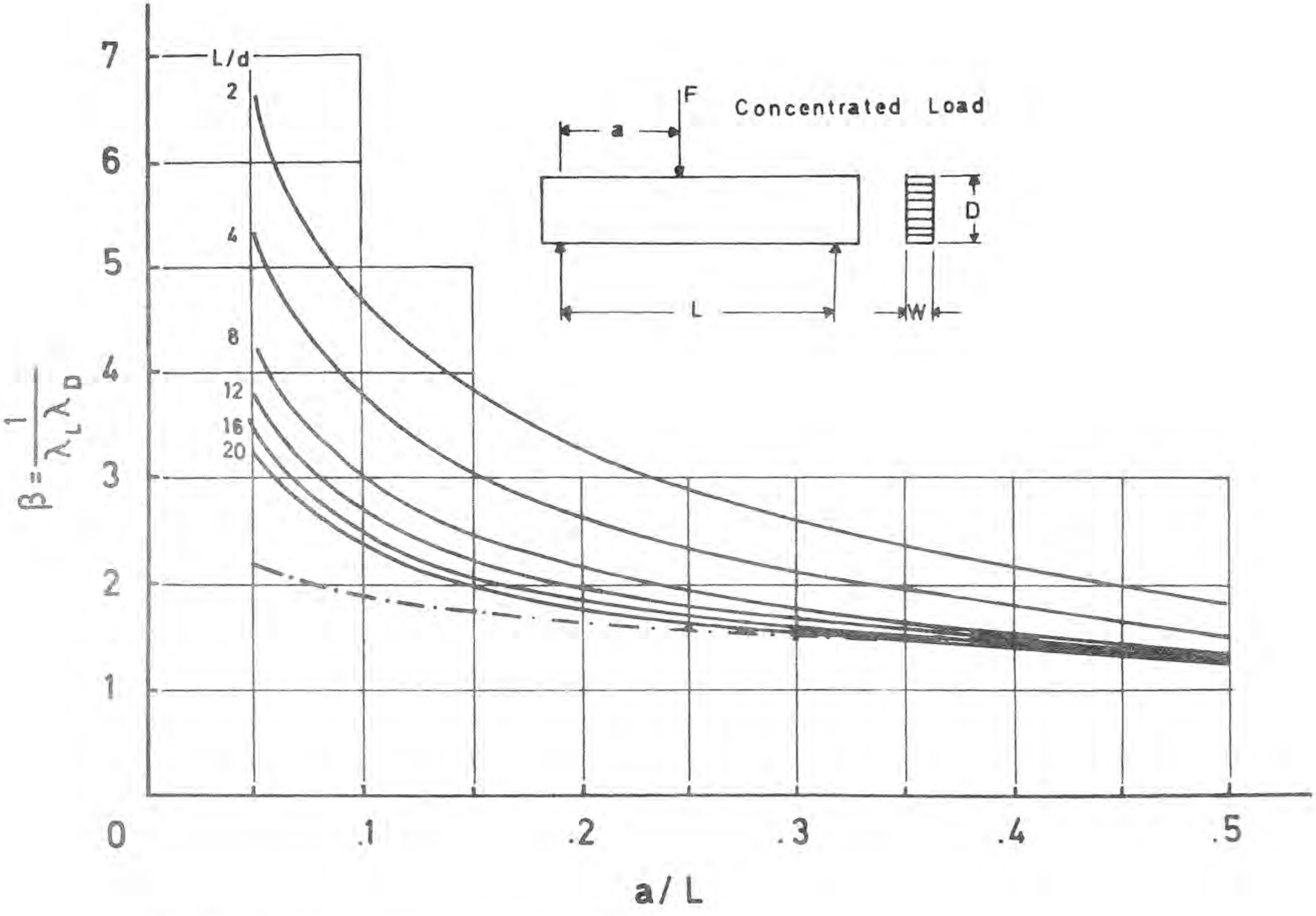


fig.5: β-values for concentrated load

——— according to Foschi/Barrett [4]

—·— according to section 2

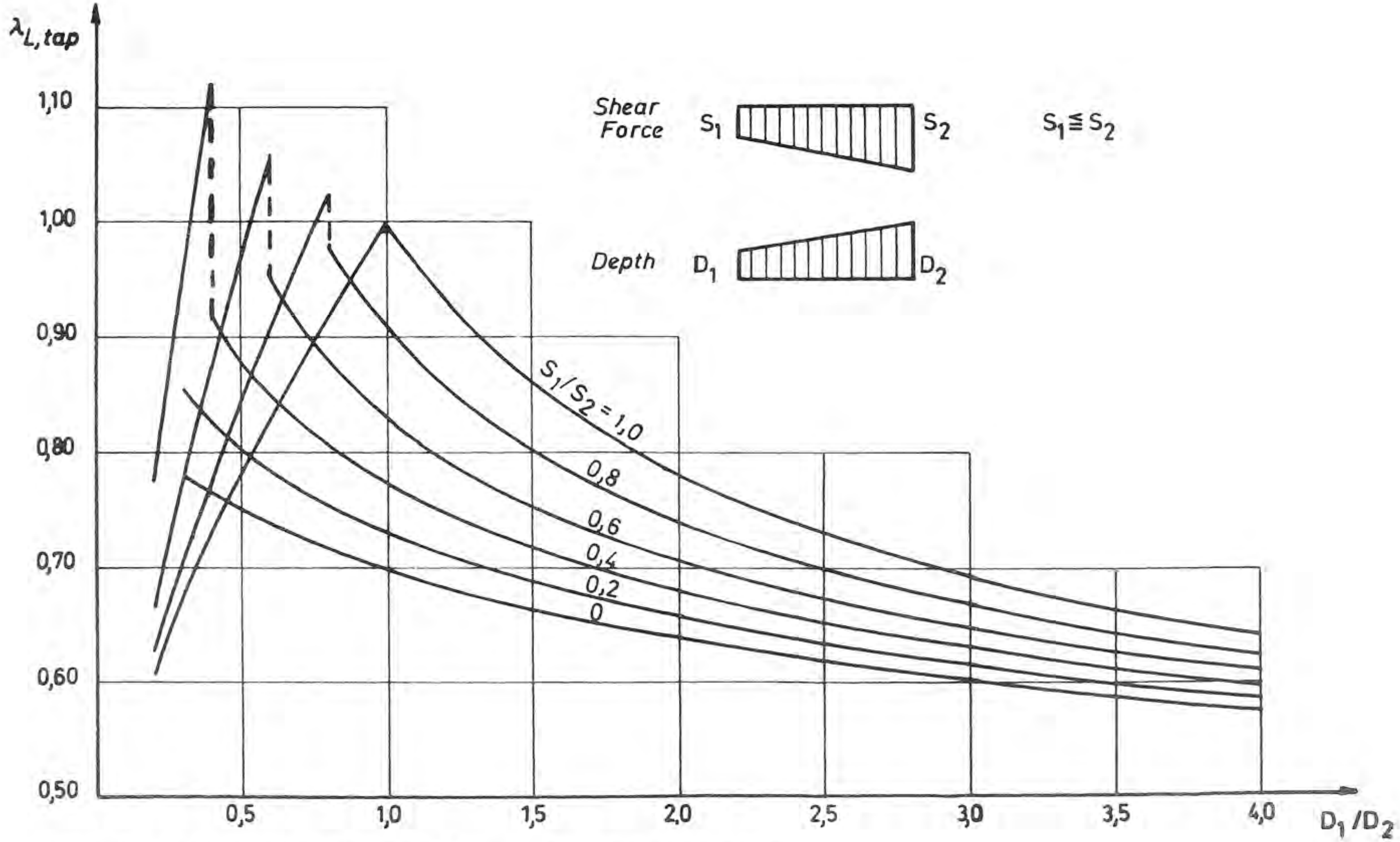


fig.6: fullness-parameter $\lambda_{L,tap}$ for tapered beams