The Paradox of Tristram Shandy[1]

W. Mueckenheim

University of Applied Sciences, Augsburg

It is shown that set theory and mathematics yield different limits for a special sequence of sets.

For more then 100 years mathematicians have been using, along with the potentially infinite, represented by the never ending sequence of natural numbers 1, 2, 3, ..., the finished infinite, represented by the finished set of all natural numbers. This application goes back to Georg Cantor, who transferred the bijection as a mathematical measure from the finite domain into the infinite. If a servant has to arrange the table for many guests, he need not count knifes and forkes separately. If for every knife there is a fork and for every fork there is a knife, then he knows that there are as many knifes and forkes. Cantor enumerated all fractions by the natural numbers and concluded that there are as many fractions as natural numbers. Infinite sets that can be enumerated he called countably infinite.

His idea necessarily presupposes the existence of all natural numbers, because otherwise there was no comparison possible. When Dagobert Duck receives a sack of Dollars, how many does he have to count in order to know the exact sum? All, of course. How man words of a message have to be received in order to understand the message? All, of course, including the end signal. Otherwise there could follow a negation.

Carl Friedrich Gauss had strictly dismissed the idea of the completed infinite: "first of all I protest against the use of an infinite magnitude as a completed one, which in mathematics is never allowed" (Gauss to Schumacher, July 12, 1831 [1]). But Cantor was not impressed. He trusted in St. Augustin [2] "all finite cardinal numbers are present distinctly and simultaneously in the divine mind. ... They form in their collection a diverse uniform thing that is separated from all the other contents of the divine intellect which again is itself a subject of divine recognition. (Cantor to Jeiler, 20. 5. 1888 [3]). Based upon that he believed to be able to contradict Gauss: "the erroneous in that phrase of Gauss consists in his statement that the completed infinite could not be subject of mathematical reflection" (Cantor to Lipschitz. 19. 11. 1883 [4]).

---

[1]) English translation of:
W. Mückenheim: "Das Paradoxon des Tristram Shandy", Research Report 2012, University of Applied Sciences Augsburg, wmm wirtschaftsverlag, Augsburg (2012) p. 242-244.
http://www.hs-augsburg.de/medium/download/oeffentlichkeitsarbeit/publikationen/forschungsbericht_2012.pdf

The completed infinite not only leads to such strange sentences as that the infinite is finished or that there are exactly as many fractions as integers, although between two integers there are always infinitely many fractions, but raises also the paradox of Tristram Shandy [5] that Adolf Fraenkel tells us in order to explain set theory [6]: "Well known is the story of Tristram Shandy who is going to write his biography, in fact so pedantically that he needs one year to complete a single day of his life. Of course his biography will never get ready, when continuing in this manner. But if he lived infinitely long (a countable infinity of years, say) then his biography would get 'ready', every day of his life, how late ever, finally would get a description.

Tristram Shandy borrows time in order to catch up with time already spent. (Similarities with modern finance schemes are unmistakable.)

But we will show that this method leads to a strong contradiction. For that sake we use the less spectacular but better understandable ratio of 2 to 1. In an urn we fill successively pairs of consecutively enumerated marbles. Between every two steps of filling we remove the marble with the lowest number from the urn. If the numbers contained in the urn are separated by a point from those removed already, we get the following sequence

```
  21.
   2.1
 432.1
  43.21
6543.21
 654.321
   ...
```

which however gets confusing when multi-digit numbers get involved. The picture gets clearer, when the even numbers are denoted by 0 and odd numbers by 1.

```
  01.
   0.1
 010.1
  01.01
0101.01
 010.101
   ...
```

After infinitely many exchanges we have a contradiction between set theory and mathematics: Set theory yields as the limit the empty urn because for every marble the step can be determined, when it is removed. Of course this result can also be formalized, compare, for instance, limes inferior und limes superior of sequences of sets [7]).

But the sequence can also be understood as a sequence of real numbers. Mathematics shows that the (improper) limit of the continued fraction

$$\cfrac{\cfrac{\cfrac{\cfrac{10^0}{10}+10^1}{10}+10^2}{10}+10^3}{10}+\ldots$$

is infinite. And since the logarithm is a strictly increasing function of its argument, we have a contradiction in that, according to mathematics, the urn will contain infinitely many marbles in the limit.

As one and the same sequence of sets leads to two different results in set theory and in mathematics, set theory can no longer be assumed as the basis of mathematics. In particular the results of a scientific theory must not depend on the choice of symbols. But even that occurs in set theory. If we let the above thought-experiment run materially exactly as before, but remove always the largest number instead of the smallest, then the urn contains infinitely many marbles in the limit. The question what might happen when an accidental number is removed remains without answer.

How could such a contradiction appear? The true reason is the identification of *all* and *each* that is based upon and suggested by the use of the same symbol $\forall$ in logic. A bijection between natural numbers and a countable set enumerates *each* element like a street number is fixed on a house. But that means only for finite sets that *all* elements are enumerated*. In infinite sets there is for *each* enumerated element an infinity of not enumerated elements - and this relation does *never* change. Unfortunately in set theory "never" is mistaken for a time that can be arrived at. Just like every infinity can be completed, so can eternity as an infinite set of seconds.

In addition a quantifier exchange has occured. The correct sentence "for every initial segment (1, 2, 3, ..., *n*) of the natural numbers there is a larger one" has been inverted to "there is an initial segment of the natural numbers (namely the complete set of all natural numbers) that is larger than every other initial segment". Actually mathematicians, strictly committed to logic, should have refused to be taken for a ride. Probably nobody would wish to invert the sentence "every citizen of Augsburg lives in Bavaria" into "every citizen of Bavaria lives in Augsburg".

---

*) Classical logic was abstracted from the mathematics of finite sets and their subsets. ... Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and Original sin of set theory [8].

References:

[1] Königliche Gesellschaft der Wissenschaften zu Göttingen: "Carl Friedrich Gauß – Werke", Band VIII, Olms, Hildesheim 1981, p. 215.
[2] Augustinus: "De Civitate Dei", XII. Buch, Kap. 19.
[3] C.Tapp: "Kardinalität und Kardinäle – Wissenschaftshistorische Aufarbeitung des Briefwechsels zwischen Georg Cantor und katholischen Theologen seiner Zeit", Steiner, Stuttgart 2005, p. 415.
[4] H. Meschkowski, W. Nilson: "Georg Cantor – Briefe", Springer, Berlin 1991, p. 148.
[5] L. Sterne: "The Life and Opinions of Tristram Shandy", Ward, Dodsley, Becket & DeHondt, 1759–1767.
[6] A.A. Fraenkel: "Einleitung in die Mengenlehre" 3. Aufl., Springer, Berlin 1928, p. 24.
[7] G. Walz et al.: "Lexikon der Mathematik", Spektrum, Heidelberg 2003. http://en.wikipedia.org/wiki/Limit_superior_and_limit_inferior
[8] H. Weyl: "Mathematics and logic", American Mathematical Monthly 53, 1946, p. 2.