Countability Contradicted

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Consider the matrix of all positive fractions m/n

1/1, 1/2, 1/3, 1/4, ...
2/1, 2/2, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ...
4/1, 4/2, 4/3, 4/4, ...
5/1, 5/2, 5/3, 5/4, ...
...

index them, following Cantor's prescription, by indices $k = (m + n - 1)(m + n - 2)/2 + m$, and gather them in the first column $(k, 1)$. Then all other columns of the matrix are empty

1/1, __, __, __, ...
1/2, __, __, __, ...
2/1, __, __, __, ...
1/3, __, __, __, ...
2/2, __, __, __, ...
...

It should also be possible to model this configuration by exchanging pairs of fractions as long as the sequence in the first column evolves according to Cantor's prescription

1/1, **2/1**, 1/3, 1/4, ...	1/1, **3/1**, 1/3, 1/4, ...	1/1, 3/1, **4/1**, 1/4, ...	1/1, 3/1, 4/1, 1/4, ...
1/2, 2/2, 2/3, 2/4, ...	1/2, 2/2, 2/3, 2/4, ...	1/2, 2/2, 2/3, 2/4, ...	1/2, **5/1**, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ...	**2/1**, 3/2, 3/3, 3/4, ...	2/1, 3/2, 3/3, 3/4, ...	2/1, 3/2, 3/3, 3/4, ...
4/1, 4/2, 4/3, 4/4, ...	4/1, 4/2, 4/3, 4/4, ...	**1/3**, 4/2, 4/3, 4/4, ...	1/3, 4/2, 4/3, 4/4, ...
5/1, 5/2, 5/3, 5/4, ...	5/1, 5/2, 5/3, 5/4, ...	5/1, 5/2, 5/3, 5/4, ...	**2/2**, 5/2, 5/3, 5/4, ...
...	...	...	...

but it is not. The first column contains as many places as before but every exchange leaves all positions of the matrix occupied. There is no definable term of the sequence of configurations which shows an empty place for the first time. If clearing happens, then not by definable terms.

What causes this failure? It is impossible to empty the matrix. This holds also in the first case, but it is not as obvious there as in the second case. If the matrix really became empty by well-ordered, indexed terms, then one of its columns would be the first to get empty. Of course this is not possible, but this necessity is often denied by switching to potential infinity which has no last elements. By exchanges of matrix-elements however it becomes obvious that not even any single position can be cleared in a definable way, let alone all positions. Countability is a tempting but impossible concept.