Dark Numbers in Set Theory

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Abstract: It is shown that there exist natural numbers, so-called dark numbers, that have no decimal representation and cannot be specified otherwise.

A natural number n is defined as a positive (sometimes also as a non-negative) integer that has a finite specification, often given by a word like "seven", sometimes by a puzzle like "how many coins do I have in my pocket?", but usually given by a finite sequence of digits. The set $\mathbb{N}$ of natural numbers n is defined as the set that contains all these natural numbers as elements and nothing else. It will be shown however that the set $\mathbb{N}$, if it is actually infinite, i.e., if it is larger than all finite sets of natural numbers, also must include elements that cannot be specified.

Every natural number n that can be specified has only a finite set of predecessors and an infinite set of successors. n divides the set $\mathbb{N}$ (well-ordered by its natural order "larger than") into two subsets, the Finite Initial Segment (FIS) $F_n = \{1, 2, 3, ..., n\}$ of n, and its complement, the endsegment $\{n+1, n+2, n+3, ...\}$. The complement is always infinite and therefore larger than any FIS. As the endsegments are sets of natural numbers, it is of interest to find out what kind of numbers is concerned.

For this sake we consider the set $\mathbb{F} = \{F_n | n \in \mathbb{N}\}$ of all FIS and observe that its union

(1) $F_1 \cup F_2 \cup F_3 \cup ... = \mathbb{N}$

does not change when one of the FIS, say F_n, is omitted:

(2) $F_1 \cup F_2 \cup F_3 \cup ... \cup F_{n-1} \cup F_{n+1} \cup F_{n+2} \cup F_{n+3} \cup ... = \mathbb{N}$.

Since every FIS can be omitted in (1) without changing the result $\mathbb{N}$, the set of FIS that can be omitted is the set $\mathbb{F}$ of all FIS existing; there is no FIS whose omission in (1) would change the result of the union.

Further we observe that in (2) the FIS which are predecessors, i.e., subsets, of the omitted F_n can be omitted too with no effect so that we get

(3) $F_{n+1} \cup F_{n+2} \cup F_{n+3} \cup ... = \mathbb{N}$

and since the process of omission does never stop, every F_n can be omitted without any effect so that we get finally

(4) $\bigcup\{\} = \{\} = \mathbb{N}$.

Of course, this is a false result since $\mathbb{N}$ is not empty. But the usual explanation, that every FIS can be omitted only as long as there remain larger FIS, does not hold, firstly because this condition is not required in (2) since a FISON without successor does not exist and therefore cannot be omitted, and secondly because we can see that as long as F_n is a proper subset of $\mathbb{N}$, it is neither necessary nor sufficient to yield the union $\mathbb{N}$. Alas, this holds for all FIS how large they ever may be. Therefore we must accept the implication:

(5) $F_1 \cup F_2 \cup F_3 \cup ... = \mathbb{N} \Rightarrow \{\} = \mathbb{N}$.

In words: **if** $\mathbb{N}$ is the union of all FIS, **then** $\mathbb{N}$ is empty. By contraposition, it follows that $\mathbb{N}$ is not the union only of all FIS of natural numbers that have specifications.

What remains? Numbers that cannot be specified: Dark numbers.

Note: The implication (5) only holds if $\mathbb{N}$ is presumed to be actually infinite, i.e., larger than every FIS, because only then every FIS is too small to be relevant for the union. In case of a potentially infinite view $\mathbb{N}$ is not a fixed set but the temporary maximum F_m of an ever increasing sequence of FIS. In this view not all FIS exist simultaneously and therefore not all FIS of the union can be omitted without changing the result of

(6) $F_1 \cup F_2 \cup F_3 \cup ... \cup F_m = \mathbb{N}$.

This view is supported by the sequence of all gapless unions of FIS (the terms are written, in two versions, below each other for clarity):

$\{1\} = \{1\}$
$\{1\} \cup \{1, 2\} = \{1, 2\}$
$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} = \{1, 2, 3\}$
...

All unions are finite. But the terms of the sequence contain, as elements of FIS, all natural numbers that can be specified, and all FIS (on the right-hand side), and also all gapless unions of these FIS (on the left-hand side). An infinite union that is larger than all FIS is impossible because there are no further FIS that could be unioned in order to surpass themselves.

Conclusion: If $\mathbb{N}$ is larger than all finite unions of FIS, then it must contain something larger than all FIS of natural numbers that have finite specifications. This something can only consist of so-called dark numbers. Since no FIS but every endsegment is infinite, almost all elements of $\mathbb{N}$ are dark numbers.